Family Floer mirror space for local SYZ singularities

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Goal: study SYZ singularities by family Floer   **Slogan:** the dual singular fibration has an elementary formula

Consider a Lagrangian fibration with singularities

$$\pi : X \to B$$

on a Kähler manifold (X, ω). Denote its smooth part by

$$\pi_0 : X_0 \to B_0$$

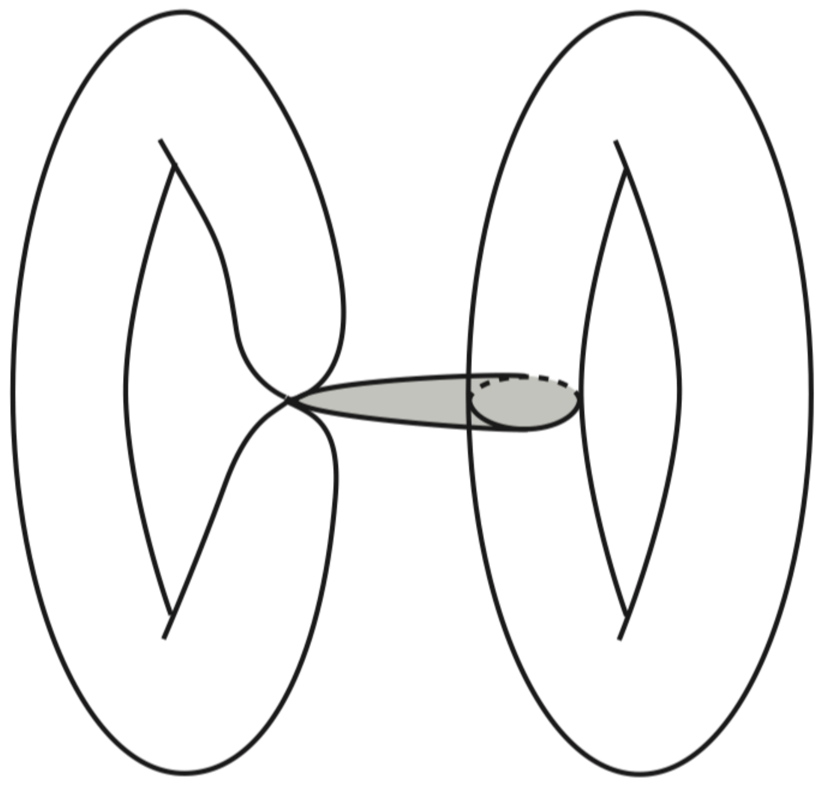
But, we must place π_0 in X not just in X_0, because the holomorphic disks we consider are all sweeping in X

Semipositive: The Maslov indices ≥ 0
(sufficient conditions: special / graded Lagrangians)

Weak Unobstructedness: for simplicity, consider a sufficient condition:
the π-fibers are preserved by an anti-symplectic involution φ

- e.g. complex conjugate $z_i \mapsto \bar{z}_i$ for Gross's Lagrangian fibration
- Then, due to the work of Solomon, the Maslov-0 terms in the Maurer-Cartan equations will be cancelled pairwise : $\beta \leftrightarrow -\varphi_* \beta$ in $\pi_2(X, L)$
- This never means Maslov-0 counts vanish. They still contribute to both the wall-crossing and the homological perturbations of A_∞ algebras.

For the singular Lagrangian fibers, we study two different types of holomorphic disks



Type (I)

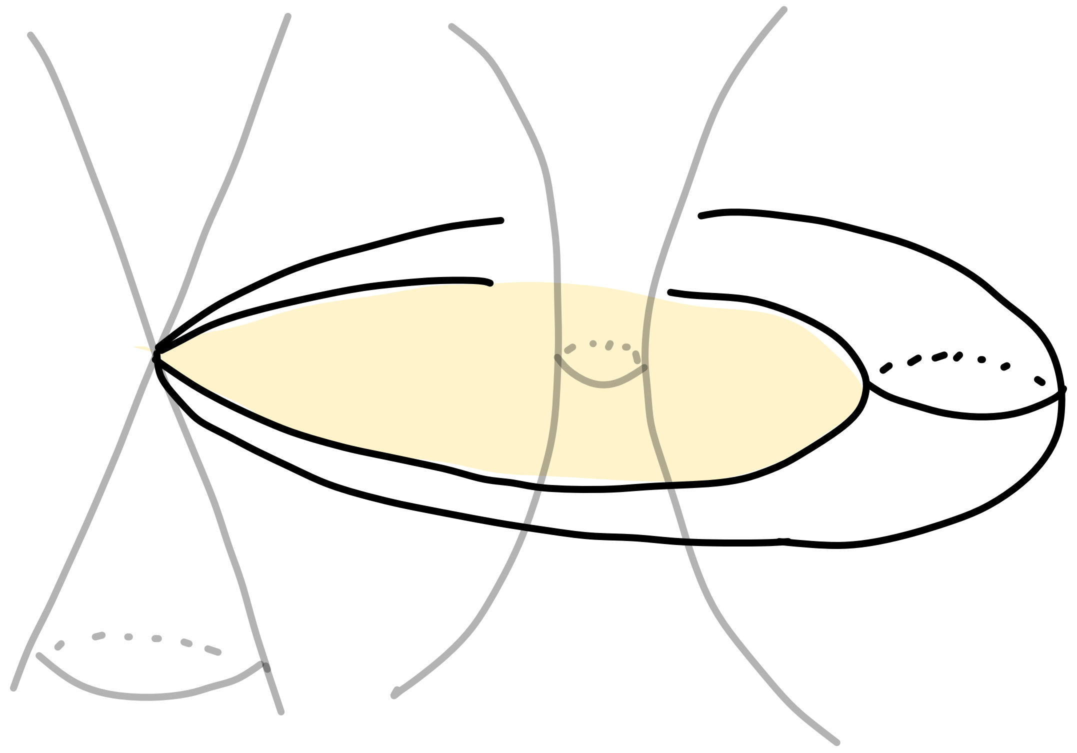
(I) Disks emanating from various singular fibers but eventually bounded by a smooth fiber

(II) Disks bounded by a singular fiber

Alternatively, the disks meet the singular fibers at interior vs boundary

◉ We must deal with them in two different ways, respectively emphasizing

- the Floer aspect for (I) (Left figure: Done in my thesis)
- the NA analytic / topological aspect for (II) (Right figure: discuss today)



Type (II)

Let's first briefly review the mirror construction in my thesis

Theorem (Y.)

We can associate to (X, π_0) a triple $(X_0^\vee, \pi_0^\vee, W_0^\vee)$ consisting of

(a) a Λ-analytic space $X_0^\vee$

(b) an affinoid torus fibration $\pi_0^\vee : X_0^\vee \to B_0$

(c) a global function $W_0^\vee$

unique up to isomorphism of analytic spaces

$\Lambda = \mathbb{C}((T^{\mathbb{R}}))$ - **Novikov field** - NA valuation v or norm $|z| = e^{-\mathrm{v}(z)}$

We also set $\Lambda_0 = \{|z| \leq 1\}$, $\quad \Lambda_+ = \{|z| < 1\}$

$U_\Lambda = \{|z| = 1\}$, Novikov unitary group, similar to $U(1) \cong S^1$

- Set-theoretically, the mirror space is not very interesting:

$$X_0^\vee = \bigcup_{q \in B_0} H^1(L_q; U_\Lambda)$$

- Then, a main point in my thesis is that on this set, we can further give an analytic space structure by considering the Maslov-0 disks.
- This <u>analytic topology</u> on $X_0^\vee$ already contains (partial) information of singularities. These disks usually meet the singular fibers.

<u>Affinoid torus fibration:</u>

It is simply a continuous map with respect to <u>analytic topology</u> and the <u>manifold topology</u> on B_0, and it is locally modeled on the *tropicalization map*

$$\mathfrak{trop} : (\Lambda^*)^n \to \mathbb{R}^n \qquad y_i \mapsto \mathrm{v}(y_i)$$

I think it is first introduced by Kontsevich-Soibelman. It is further studied and is given this name by Nicaise-Xu-Yu. Here 'continuous' is really a strong condition, since we use the analytic topology.

- A brief picture of the mirror is as follows:
- In B_0, let $\chi : (U, q_0) \to (V, c) \subset \mathbb{R}^n$ be a (pointed) integral affine chart. We allow $q_0 \notin U$. Then, we have an **affinoid tropical chart**

$$\begin{tikzcd} \bigcup_{q \in U} H^1(L_q; U_\Lambda) \overset{\text{set}}{=\!=} (\pi_0^\vee)^{-1}(U) \arrow[r, "\cong"] \arrow[d] & \mathrm{trop}^{-1}(V-c) \arrow[d] \\ U \arrow[r, "\chi"] & V \end{tikzcd}$$

- Given two such affinoid tropical charts for $i = 1,2$

$$\tau_i : (\pi_0^\vee)^{-1}(U) \to \mathfrak{trop}^{-1}(V_i - c_i)$$

There is a transition map (or say a gluing map)

$$\Phi : \mathrm{trop}^{-1}(V_1 - c_1) \to \mathrm{trop}^{-1}(V_2 - c_2)$$

between the two *analytic open domains* in $(\Lambda^*)^n$. It is decided by **some** A_∞ homomorphism associated to an isotopy from L_{q_1} to L_{q_2} (roughly). But, Φ is the same for **any** A_∞ homomorphism obtained in this way. This is carefully proved in my thesis.

Theorem (Y.)

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unique up to isomorphism of analytic spaces

- We aim to develop an analytic extension $\pi^\vee$ over B (rather that just B_0)
- All possible (analytic) continuous extension may be too much.
- We add extra conditions to control the extension.
- Based on our computations for the Gross's special Lagrangian fibration, we propose to use:

tropically continuous maps:

in the sense of Chambert-Loir and Ducros. See Section (3.1.6) of their famous paper in which they develop (p,q)-forms on analytic space:

Formes différentielles réelles et courants sur les espaces de Berkovich

- Under this condition, the topological extension from B_0 to B can somehow control the analytic extension from $\pi_0^\vee$ to some potential extension.
- I'm inspired by Gross's Topological Mirror Symmetry to think like this. Also, I'm inspired by Kontsevich-Soibelman's singular model in

('Affine structures and non-archimedean analytic spaces', Section 8)

Affinoid torus fibration:

It is simply a continuous map with respect to analytic topology and the manifold topology on B_0 , and it is locally modeled on the *tropicalization map*

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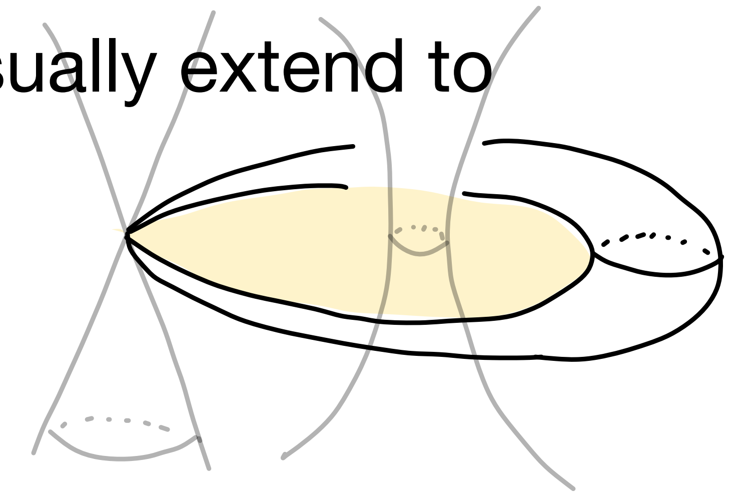
- Roughly speaking, a tropically continuous map F is *locally* in the following form:

$$F|_{\mathcal{U}} = \varphi(\mathsf{v}(f_1), \ldots, \mathsf{v}(f_n))$$

  - $\mathcal{U}$ is an analytic open subset
  - $f_1, \ldots, f_n : \mathcal{U} \to \Lambda^*$ are invertible analytic functions. (e.g. local coordinates for analytification of an algebraic variety)
  - $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is just a continuous map for Euclidean topology

- Let's give a naive idea, we will be more specific soon.
- Very intuitively, imagine $\mathsf{v}(f_1), \ldots, \mathsf{v}(f_n)$ are 'action coordinates', so more or less correspond to symplectic areas.
- The symplectic areas, as functions on B_0, can usually extend to the singular locus. See the right side figure:
- Anyway, we will just focus on an example later.



Why the conventional Maurer-Cartan idea is not enough

It's been long expected that the mirror space should be the union of the Maurer-Cartan sets (with singular fibers)

$$\bigcup_{q\in B} \mathscr{MC}(L_q)$$

Over B_0, this roughly gives the correct picture. Indeed, the $\mathscr{MC}(L_q)$ is very close to $H^1(L_q; U_\Lambda)$ (slightly different)

In short, the **set-theoretic cocycle condition** is essentially straightforward by the homotopy invariance of Maurer-Cartan *sets.*

(a property well-known for the classic homotopy theory of A_∞ structures).

Over $B\backslash B_0$, the Maurer-Cartan picture fails unfortunately !

Maurer-Cartan set of a singular Lagrangian

$\subsetneq$ **'Dual singular fiber'**

Roughly, the 'dual singular fibers' here will extend $\pi_0^\vee$ *tropically continuously*. We will see this more clearly soon.

Admittedly, the Maurer-Cartan picture may offer some good ideas, or inspirations, etc.

But unfortunately, it is eventually not the correct picture.

Nevertheless, for the **analytic cocycle condition**, we must introduce more powerful ideas and tools as in my thesis: (at least 4 points below)

1. Study a uniform version of *Groman-Solomon's reverse isoperimetric inequalities* for the non-archimedean convergence.
2. Establish a minimal model version of Fukaya's trick. With nontrivial Maslov-0 disks, this creates further difficulties. Why minimal model? Very roughly, want $H^1(L_q; U_\Lambda)$ rather than $\Omega^1(L_q; U_\Lambda)$.
3. Prove the transition maps are well-defined. Otherwise, what the cocycle conditions mean is very ambiguous. This requires the following ud-homotopy.
4. Upgrade the classic homotopy to the **ud-homotopy** for the A_∞ structures. The point is, the analytic gluing needs stronger homotopy.
   - For example, individual A_∞ maps satisfy the divisor axiom are not enough. We want the homtopies between them also satisfy the divisor axiom in a very specific sense.
   - This requires lots of difficult homological algebras, and finally we need to introduce the so-called category $\mathscr{UD}$ in my thesis.

In a word, the ud-homotopy theory enables us to **upgrade the 'classic Maurer-Cartan idea' to a higher and more precise level**, matching NA adic-convergent formal power series rather than just set bijections.
This is a totally different story, and is crucial for the analytic topology.

❖ By the way, the inclusion of Landau-Ginzburg superpotential will be also indispensable for our results later.

Theorem: Y is SYZ mirror to X

Let's go to a fundamental example which has been long predicted by Gross-Siebert program. Define

$$X = \mathbb{C}^n \backslash \{z_1 \cdots z_n = 1\}$$

(equipped with the standard symplectic form), and define

$$Y = \{(x, y) \in \Lambda^2 \times (\Lambda^*)^{n-1} \mid x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$$

Definition: We say an algebraic variety Y over Λ is **SYZ mirror** to a complex manifold X over $\mathbb{C}$ if

- there exists a proper tropically continuous analytic fibration $f : \mathscr{Y} \to B$ on a Zariski-dense analytic open domain $\mathscr{Y}$ in Y
- there exists a Lagrangian fibration $\pi : X \to B$ onto the same base manifold B for some Kähler form ω on X

such that the following conditions hold

1) The π (resp. f) restricts to a Lagrangian torus fibration π_0 (resp. an affinoid torus fibration f_0) over a common open subset $B_0 \subset B$ such that the **two** induced integral affine structures agree with each other and $\Delta = B \backslash B_0$ is codimesion-2.
(same smooth/singular locus, integral affine str. It's already very nontrivial)
2) There is an isomorphism of affinoid torus fibration $\pi_0^\vee \cong f_0$. Here $\pi_0^\vee$ is the family Floer mirror fibration for π_0 (a sort of T-duality)
3) The set $\mathscr{Y}_0 := f_0^{-1}(B_0)$ is Zariski dense in Y
(possibly redundant, but useful for a folklore conjecture later)

- Kontsevich-Soibelman proved that any affinoid torus fibration also induces an integral affine structure on the base.
- The existence of affinoid torus fibration, parallel to that of Lagrangian fibration, should be also a nontrivial problem in NA geometry.
- In general, $\mathscr{Y}$ *depends on ω, and vice versa.* This gives a picture of "Kähler moduli v.s. complex moduli": $(Y, \mathscr{Y})$ is SYZ mirror to (X, ω).
- For example, we can simply run the family Floer T-duality for the toric moment map. The analytic domain is $\mathscr{Y} = \mathscr{Y}_0 \cong \mathfrak{trop}^{-1}(P)$ in $(\Lambda^*)^n$ for the moment polytope $P = P_\omega$ relying on ω. But, the Zariski-closure is the same algebraic variety $Y = (\Lambda^*)^n$ not relying on ω.
- We focus only on SYZ now. Hopefully, we could achieve Abouzaid's family Floer functor to prove HMS in the future.

Remark *if you allow me to remove the condition 2), still nontrivial to get 1), then*

- Nothing about Floer/Fukaya theories
- Nothing about the moduli spaces of pseudo-holomorphic disks.
- It may be very unmotivated without some Floer-theoretic considerations.
- But, as we will see, the construction of f on Y itself is very elementary

$$\begin{tikzcd} X \arrow[d, "\pi"] & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"] & X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & \mathscr{Y} \arrow[d, "f"] \\ B & B_0 \arrow[l, hook'] & B_0 \arrow[r, hook] & B \end{tikzcd}$$

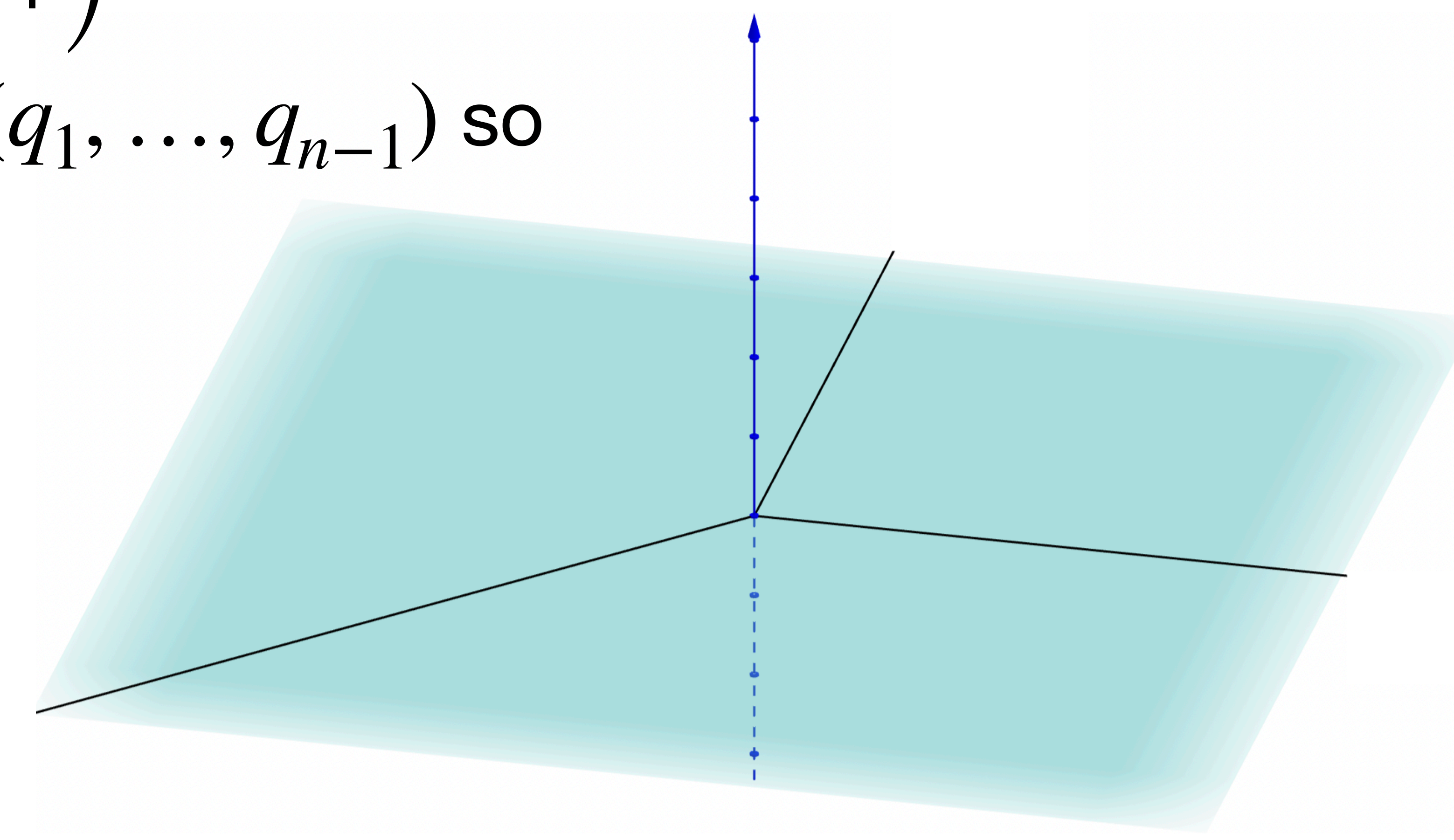
Family Floer
T-duality

Theorem: $Y = \{x_0x_1 = 1 + y_1 + \cdots + y_{n-1}\}$ is SYZ mirror to $X = \mathbb{C}^n \backslash \{z_1 \cdots z_n = 1\}$

- On the A-side, we consider a Gross's special Lagrangian fibration.

$$\pi : X \to B \equiv \mathbb{R}^n \qquad z \mapsto \Big(|z_1|^2 - |z_n|^2, \ldots, |z_{n-1}|^2 - |z_n|^2, \log|z_1 \cdots z_n - 1| \Big)$$

- The singular locus $\Delta = B \backslash B_0$ is a tropical hypersurface in $\mathbb{R}^{n-1} \times \{0\}$ given by those $\bar{q} = (q_1, \ldots, q_{n-1})$ so that $\min\{0, q_1, \ldots, q_{n-1}\}$ attains at least twice. (See the right figure when $n = 3$)



- Let $B_0 := B \backslash \Delta$. By the family Floer theory, there is an 'abstract' affinoid torus fibration

$$\pi_0^\vee : X_0^\vee \to B_0$$

on the 'abstract' set $X_0^\vee \equiv \bigcup_{q \in B_0} H^1(L_q; U_\Lambda)$

- Our Zariski-dense analytic open domain $\mathscr{Y}$ in $Y = \{x_0x_1 = 1 + y_1 + \cdots + y_{n-1}\}$ is defined by setting the NA norm $|x_1| < 1$. This is very explicit. We will explain why we choose it later.

  (It is also fine to consider $|x_1| < R$ for other $R \neq 1$ or replace x_1 by x_0. Just a convention)

Now, our major task is to find *both* the constructions of (see the diagram)

(i) the **analytic embedding** $g : X_0^\vee \to \mathscr{Y}_0$ from the abstract to the concrete

(ii) the **dual singular fibration** $f : \mathscr{Y} \to B$ so that $f_0 \cong \pi_0^\vee$ via the above g

$$\begin{tikzcd}
X \arrow[d, "\pi"] & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"'] \arrow[r, phantom, "\substack{\text{Family Floer} \\ \text{-----} \\ \text{T-duality}}"] & X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & \mathscr{Y} \arrow[d, "f"] \\
B & B_0 \arrow[l, hook'] & B_0 \arrow[r, hook] & B
\end{tikzcd}$$

(i) Construction of g: First, study the wall-crossing of π (use the family Floer theory), and we can finally show a simple **identification**:

(*) $$X_0^\vee \cong T_+ \cup T_- \,/\, \sim$$

$T_\pm$ are analytic open subdomains $\subsetneq (\Lambda^*)^n$ (wrt the action coordinates for ω)

$T_\pm$ correspond to the Clifford and Chekanov tori respectively.

the gluing relation $\sim$ can be written down explicitly.

Under the identification (*), the analytic embedding g is obtained by gluing

$$g_+ : T_+ \to Y \qquad (y_1, \dots, y_n) \mapsto \left(\frac{1}{y_n} \,,\, y_n h \,,\, y_1, \dots, y_{n-1}\right)$$

$$g_- : T_- \to Y \qquad (y_1, \dots, y_n) \mapsto \left(\frac{h}{y_n} \,,\, y_n \,,\, y_1, \dots, y_{n-1}\right)$$

where $h = 1 + y_1 + \cdots + y_{n-1}$

Anyway, the Novikov field is good for the **T-duality idea** :

More intrinsically, $T_\pm \subset \bigcup_{q \in B_0} H^1(L_q; U_\Lambda)$ as sets, and

$$(\Lambda^*)^n \ni (y_1, \dots, y_n) \leftrightarrow \left(\; T^{\chi_1} \nabla(\sigma_1), \dots, T^{\chi_n} \nabla(\sigma_n) \;\right) \;\leftrightarrow \nabla$$

where $\nabla \in H^1(L_q; U_\Lambda)$ and $\chi = (\chi_1, \dots, \chi_n)$ is the action coordinates for a basis $\{\sigma_1, \dots, \sigma_n\} \subset \pi_1(L_q)$.

Remark: *The only place we use Floer theory is the identification* (*)

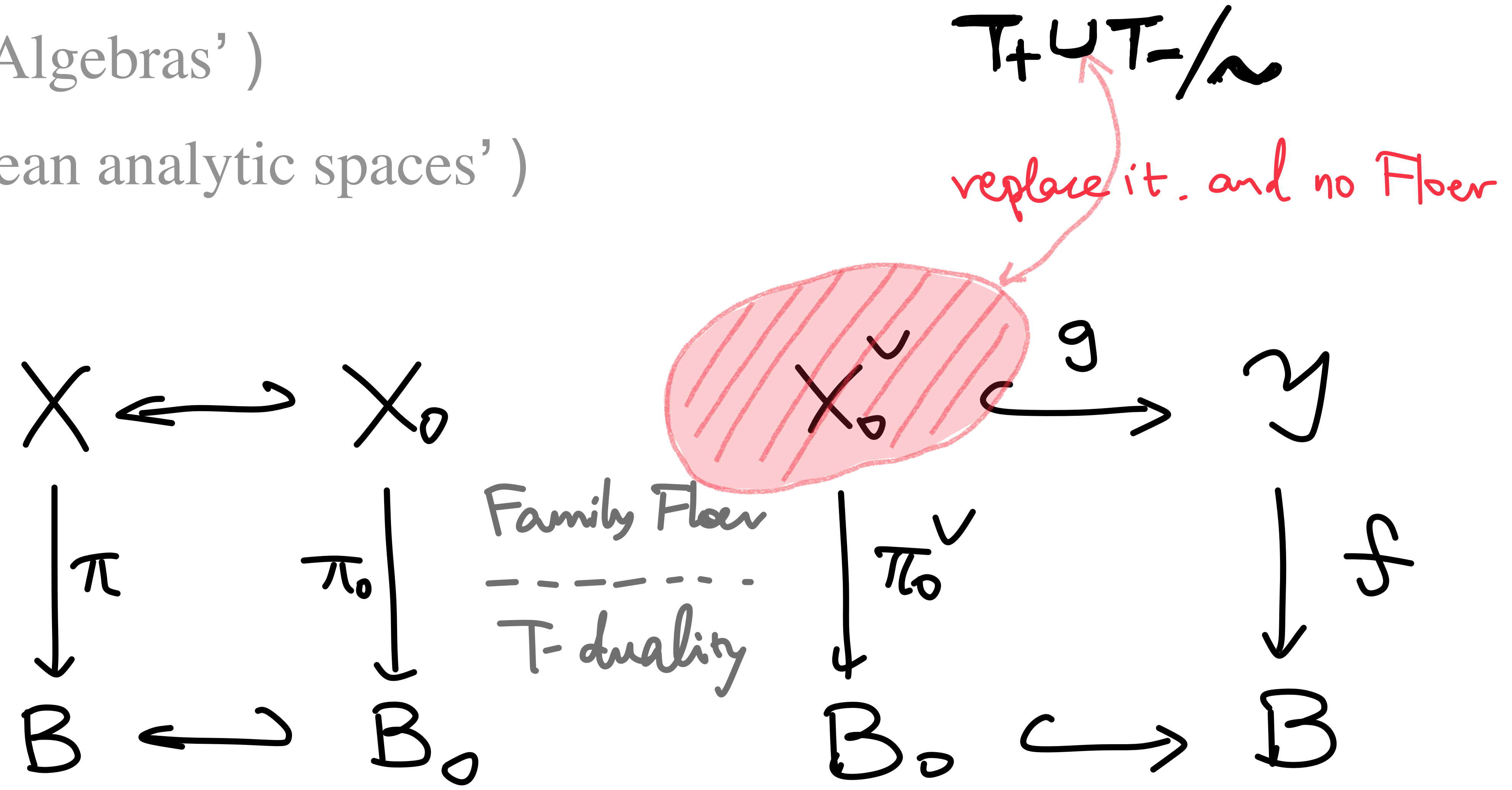
- In fact, all of $T_\pm$, the gluing relation $\sim$, the embedding g have nothing to do with the Floer theory or the moduli space business.
- If we're content with the main theorem without the T-duality condition $\pi_0^\vee \cong f_0$, then we can entirely exclude the Floer theory.

❖The formula of g is due to GHK and GS, but we further add NA picture (KS).

- Gross-Hacking-Keel (see Lemma 3.1 in 'Birational Geometry of Cluster Algebras')
- Kontsevich-Soibelman (see Page 44 in 'Affine structures and non-archimedean analytic spaces')

Because GHK is over $\mathbb{C}$, we discuss some ways of reduction from Λ to $\mathbb{C}$:

- If an analytic space over Λ is the generic fiber of a formal scheme over Λ_0, then the special fiber is the so-called *analytic reduction* which is a variety over $\mathbb{C}$. (not unique)
- $(\Lambda^*)^n$ 'contains' infinity copies of $(\mathbb{C}^*)^n$ c.f. *exploded tropicalization map* (Sam Payne).
- Study the *Maslov's dequantization*. (Mikhalkin, Abouzaid-Ganatra-Iritani-Sheridan)



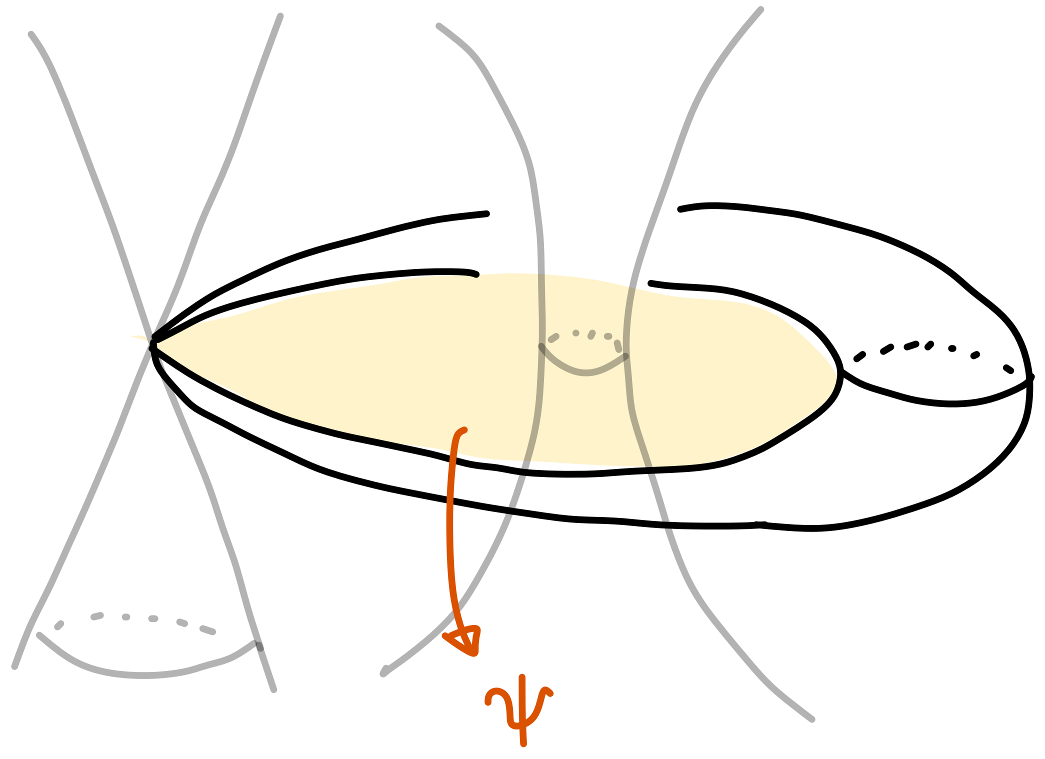
(ii) Construction of f: This is very difficult ! We obtain $f = j^{-1} \circ F$ by decomposing it into a topological embedding $j : B \equiv \mathbb{R}^n \to \mathbb{R}^{n+1}$ and an analytic tropically continuous map $F : Y \to \mathbb{R}^{n+1}$. (I will give a very beautiful picture for the image of j)

This decomposition imitates Kontsevich-Soibelman's model, and I would like to thank Tony Yue Yu for his suggestion to KS's paper.

Anyway, after a lot of trials, we find the following pair (j, F) by hand, and it finally works !

$$j(q) = (\theta_0(q), \theta_1(q), \bar{q})$$

- $q = (q_1, \ldots, q_{n-1}, q_n) = (\bar{q}, q_n)$ are in $B \equiv \mathbb{R}^n$
- $\theta_0(q) = \min\{-\psi(q), -\psi(\bar{q},0)\} + \min\{0, \bar{q}\}$
- $\theta_1(q) = \min\{\ \psi(q),\quad \psi(\bar{q},0)\}$
- $\psi : B \to \mathbb{R}_+$ is the ω-areas of holomorphic disks



$\psi(q)$ **for** $q \in \Delta \equiv B \backslash B_0$

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[d, "F"] \\ B \arrow[r, hook, "j"] & \mathbb{R}^{n+1} \end{tikzcd}$$

The Zariski-dense analytic subdomain $\mathscr{Y}$ exactly satisfies that $j(B) = F(\mathscr{Y})$. Define

$$f = j^{-1} \circ F \,|\, \mathscr{Y}$$

It is explicit, elementary, and has singular fibers

Although not obvious now, it will meet all the conditions for our definition of 'SYZ mirror' (e.g. match singular locus, integral affine str) $\square$

$$F = (F_0, F_1, G_1, \ldots, G_{n-1})$$

Given $z = (x_0, x_1, y_1, \ldots, y_{n-1})$ in Y, we define

- $F_0(z) = \min\left\{ \mathrm{v}(x_0), -\psi(\mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1}),0) + \min\{0,\mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1})\} \right\}$
- $F_1(z) = \min\left\{\ \mathrm{v}(x_1),\quad \psi(\mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1}),0)\right\}$
- $G_k(z) = \mathrm{v}(y_k) \qquad 1 \le k \le n-1$

where v is the non-archimedean valuation and the ψ is the same as above.
This depends on the Kähler form in general.
This includes 'singular' analytic fibers in the sense of Kontsevich-Soibelman.

$$\begin{tikzcd} X \arrow[d, "\pi"] & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"] \\ B & B_0 \arrow[l, hook'] \end{tikzcd} \qquad \begin{array}{c} \text{Family Floer} \\ \text{- - - - - - -} \\ \text{T-duality} \end{array} \qquad \begin{tikzcd} X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & \mathscr{Y} \arrow[d, "f"] \\ B_0 \arrow[r, hook] & B \end{tikzcd}$$

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- A prototype is found by KS many years ago in a very different context.
- KS use 3 charts, but we only use 2 charts $T_\pm$ which geometrically correspond to Clifford/Chekanov tori. We simplify it to 2 charts, exactly inspired by GHK's work.
- **As we see, the formula of f is very elementary itself (maybe still complicated)**
- In fact, if we are content with the result without the T-duality condition $\pi_0^\vee \cong f_0$ there is even no need to know anything about Floer theory

$$F = (F_0, F_1, G_1, \ldots, G_{n-1})$$

Given $z = (x_0, x_1, y_1, \ldots, y_{n-1})$ in Y, we define

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where v is the non-archimedean valuation and the ψ is the same as above.
This depends on the Kähler form in general.
This includes 'singular' analytic fibers in the sense of Kontsevich-Soibelman.

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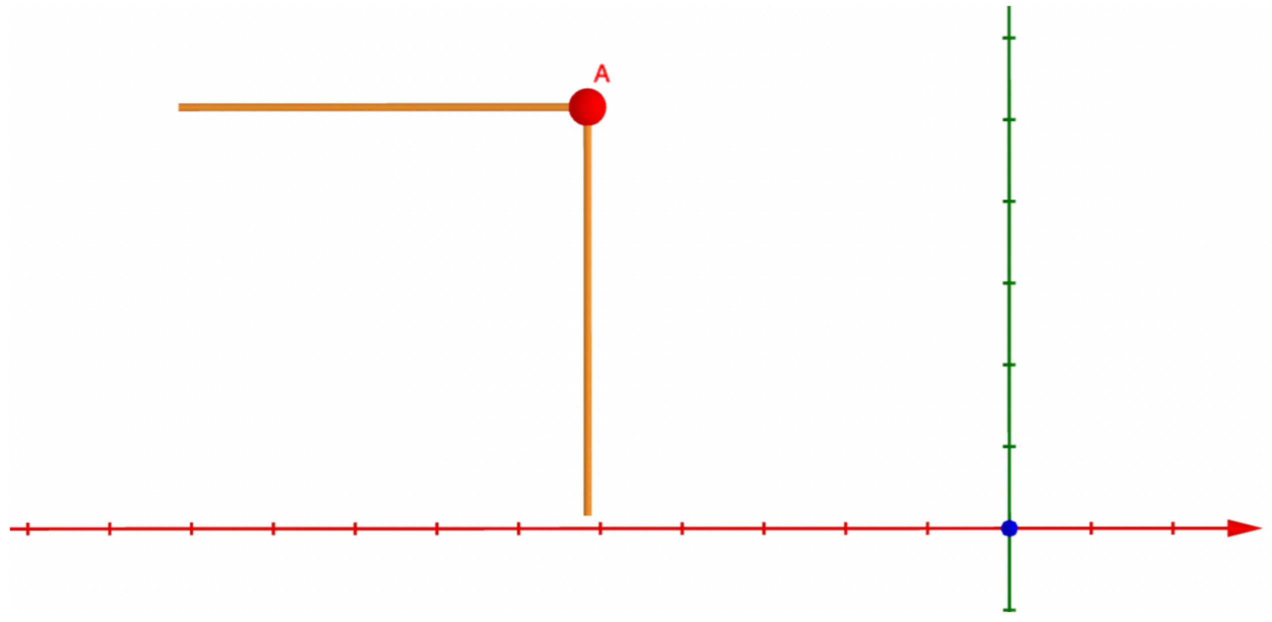
Visualization for $j : \mathbb{R}^n \to \mathbb{R}^{n+1}$

In $\mathbb{R}^2$, consider a broken line $R_{\bar{q}}$ (in orange) with a corner point :

$$A = (a_0(\bar{q}), a_1(\bar{q})) := \Big(\min\{0,\bar{q}\} - \psi(\bar{q},0) \ , \ \psi(\bar{q},0) \Big).$$

Get a family of broken lines $R_{\bar{q}}$ parametrized by $A(\bar{q})$ or $\bar{q}$.



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- $\theta_1(q) = \min\{ \ \psi(q), \quad \psi(\bar{q},0)\}$
- $\psi : B \to \mathbb{R}_+$ is the ω-area of a holomorphic disk

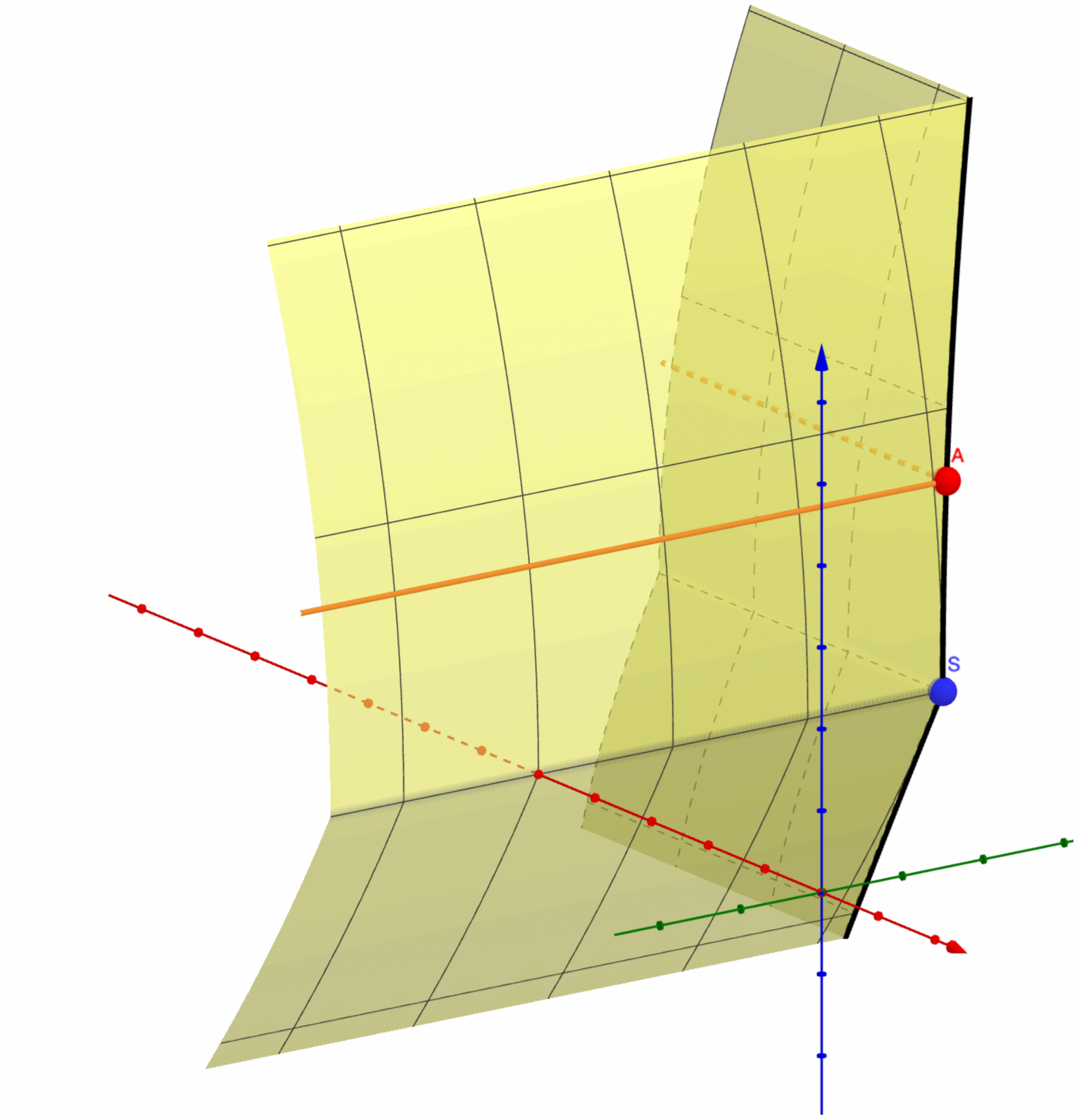
Now, the image $j(B)$ is simply the union of these broken lines:

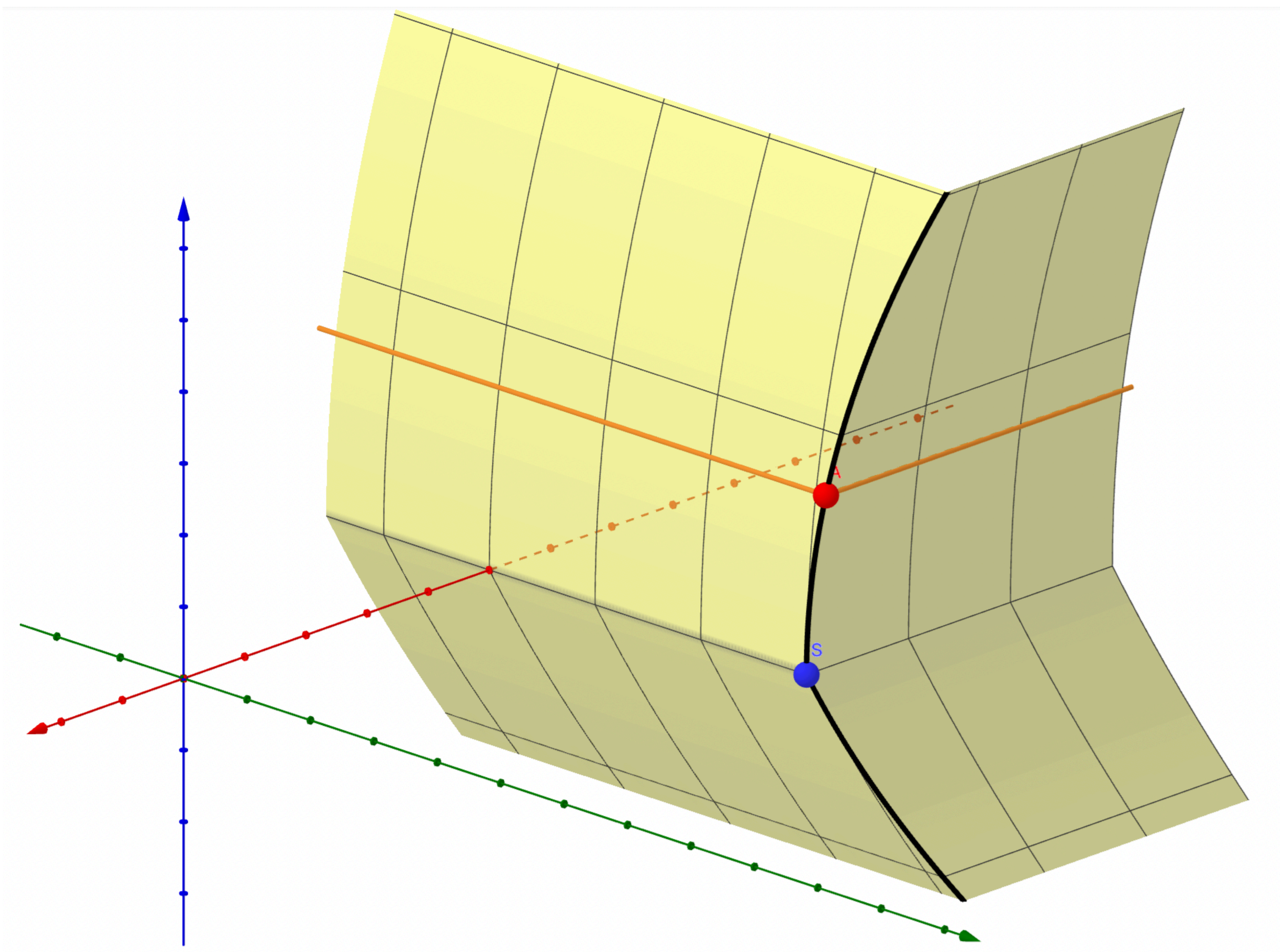
$$j(B) = \bigcup_{\bar{q} \in \mathbb{R}^{n-1}} R_{\bar{q}} \times \{\bar{q}\}$$

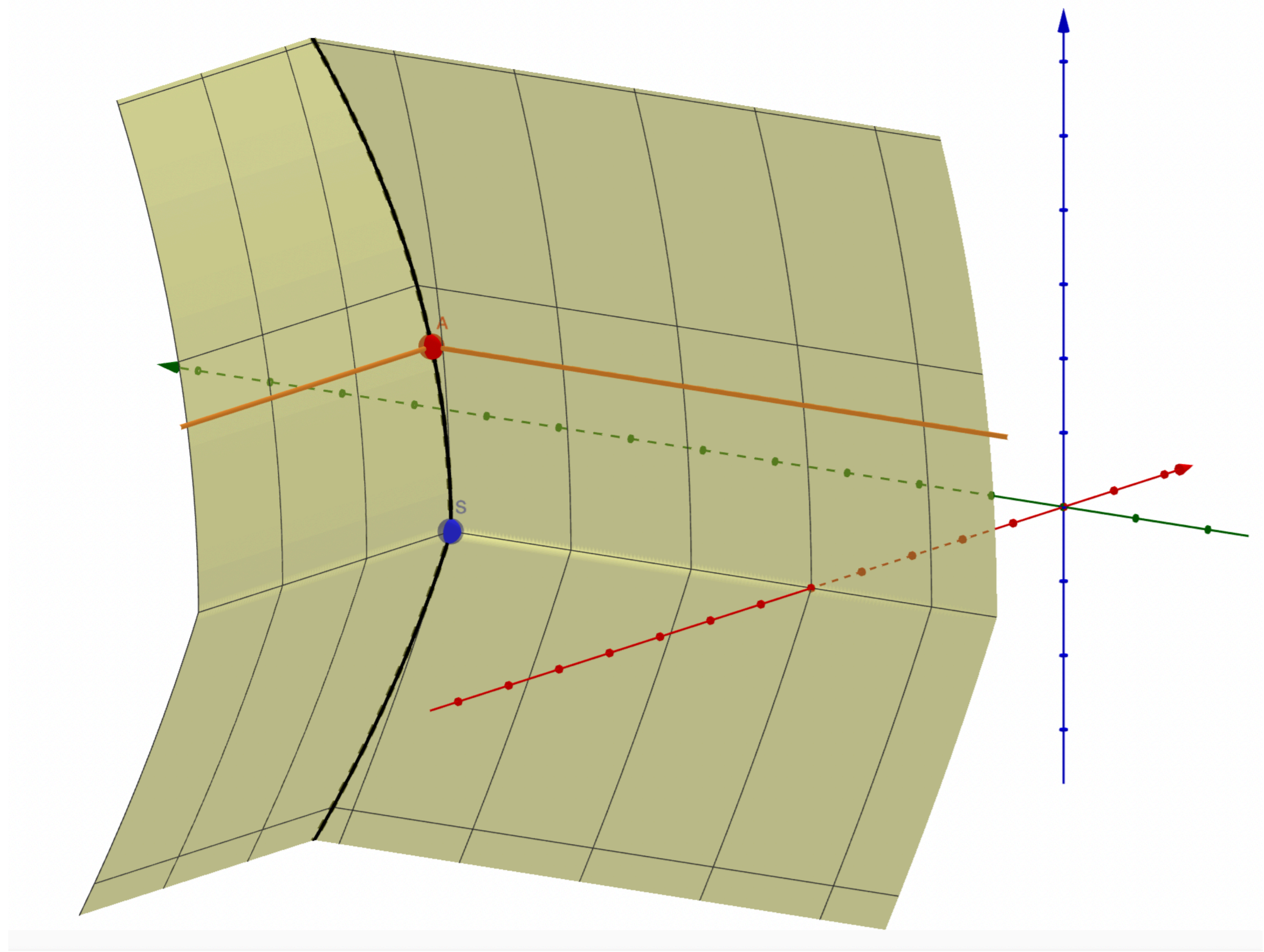
The black curve is the trace of the point $A(\bar{q})$ and relies on ω in general. Also,

$$j(\Delta) = \bigcup_{\bar{q} \in \Pi} \{(A(\bar{q}), \bar{q})\}$$

where $\Pi = V(\min\{0,\bar{q}\})$. If $n = 2$, only one singular point, the blue point below.





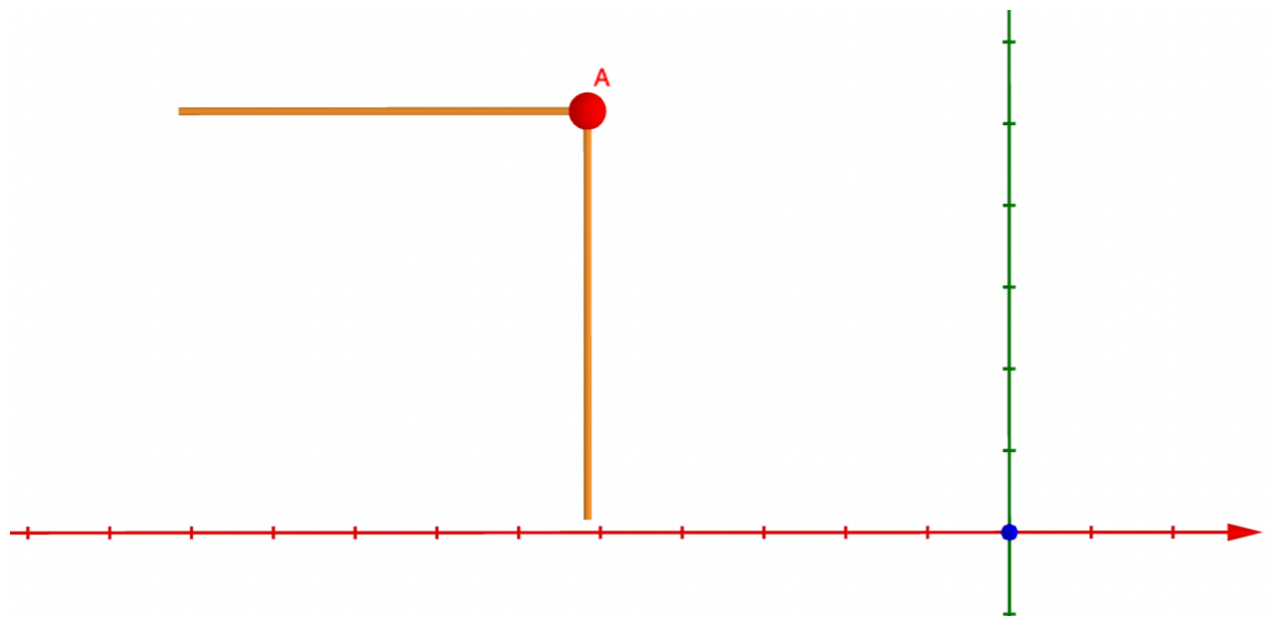


Visualization for $j : \mathbb{R}^n \to \mathbb{R}^{n+1}$

In $\mathbb{R}^2$, consider a broken line $R_{\bar q}$ (in orange) with a corner point :

$$A = (a_0(\bar q), a_1(\bar q)) := \Big(\min\{0,\bar q\} - \psi(\bar q,0)\ ,\ \psi(\bar q,0)\Big).$$

Get a family of broken lines $R_{\bar q}$ parametrized by $A(\bar q)$ or $\bar q$.



$$j(q) = (\theta_0(q), \theta_1(q), \bar q)$$

- $q = (q_1, \ldots, q_{n-1}, q_n) = (\bar q, q_n)$ are in $B \equiv \mathbb{R}^n$
- $\theta_0(q) = \min\{-\psi(q), -\psi(\bar q,0)\} + \min\{0,\ \bar q\}$
- $\theta_1(q) = \min\{\ \psi(q),\quad \psi(\bar q,0)\}$
- $\psi : B \to \mathbb{R}_+$ is the ω-area of a holomorphic disk

Now, the image $j(B)$ is simply the union of these broken lines:

$$j(B) = \bigcup_{\bar q \in \mathbb{R}^{n-1}} R_{\bar q} \times \{\bar q\}$$

The black curve is the trace of the point $A(\bar q)$ and relies on ω in general. Also,

$$j(\Delta) = \bigcup_{\bar q \in \Pi} \{(A(\bar q), \bar q)\}$$

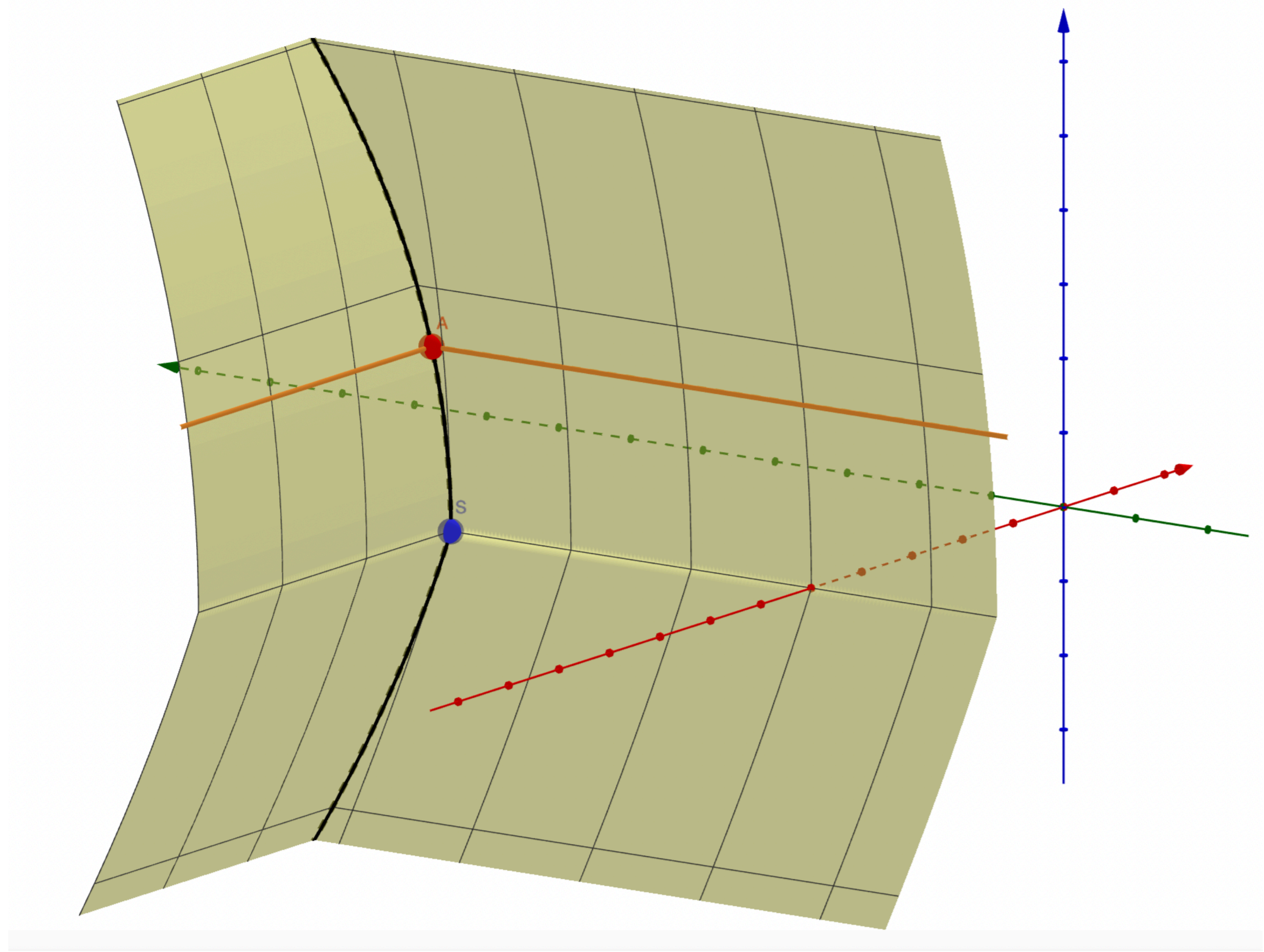
where $\Pi = V(\min\{0,\bar q\})$. If $n = 2$, only one singular point, the blue point below.

Maybe very surprisingly, the $F = (F_0, F_1, G_1, \ldots, G_{n-1})$ has **exactly** the same image in $\mathbb{R}^{n+1}$, i.e. $j(B) = F(\mathscr{Y})$

Given $z = (x_0, x_1, y_1, \ldots, y_{n-1})$ in Y, we define

- $F_0(z) = \min\Big\{ \mathrm{v}(x_0), -\psi(\mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1}),0) + \min\{0, \mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1})\}\Big\}$
- $F_1(z) = \min\big\{\ \mathrm{v}(x_1),\quad \psi(\mathrm{v}(y_1), \ldots, \mathrm{v}(y_{n-1}),0)\big\}$
- $G_k(z) = \mathrm{v}(y_k) \qquad 1 \le k \le n-1$

where v is the non-archimedean valuation and the ψ is the same as above.
This depends on the Kähler form in general.
This includes 'singular' analytic fibers in the sense of Kontsevich-Soibelman.



- The motivation to find (j, F) is difficult for me to explain. It is really found by hand.
- But, let me try to explain the idea: the key observation is

$$\mathsf{v}(1 + y_1 + \cdots + y_{n-1}) = \min\{0, \mathsf{v}(y_1), \ldots, \mathsf{v}(y_{n-1})\}$$

whenever the minimum is attained only once.

But if the minimum is attained twice, it may happen that

$$\mathsf{v}(1 + y_1 + \cdots + y_{n-1}) > \min\{0, \mathsf{v}(y_1), \ldots, \mathsf{v}(y_{n-1})\}$$

which gives some ambiguity.

- The pair (j, F) is very carefully designed to 'eliminate' this ambiguity.
- This ambiguity also plays the crucial role to 'create' the singularity of the dual fibration f.

$$\begin{tikzcd} X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"'] & Y \arrow[d, "F"] \arrow[dl, bend right, "f"'] \\ B_0 \subseteq B \arrow[r, hook, "j"] & \mathbb{R}^{n+1} \end{tikzcd}$$

- Regardless of how we find the (j, F) from the family Floer picture, it is really elementary to define them and check a key lemma as follows :

$$j \circ \pi_0^\vee = F \circ g$$

This implies the T-duality condition $\pi_0^\vee \cong f_0$

- The only place we need family Floer is the identification (not easy to get)

(*)
$$X_0^\vee \cong T_+ \cup T_- \,/ \sim$$

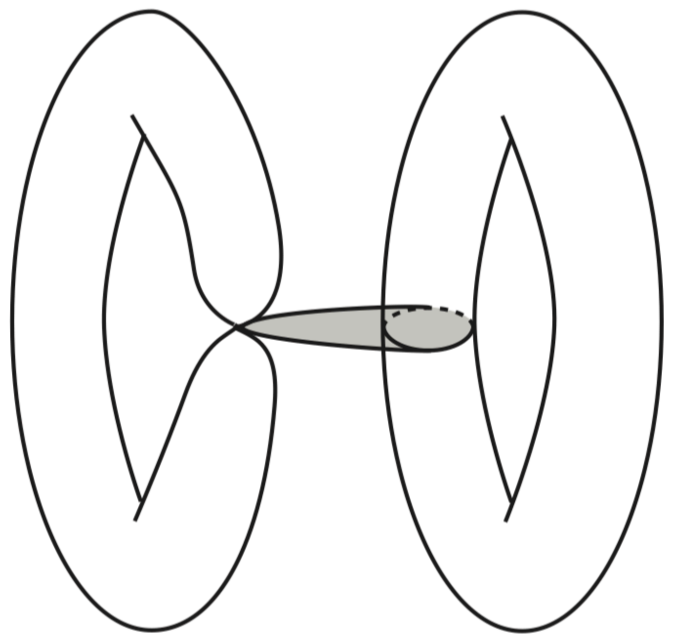
- By (*), we can also directly define the affinoid torus fibration $\pi_0^\vee$ on the domains $T_\pm$ regardless of any Floer-theoretic considerations.
- Except (*), all of $T_\pm$, $\sim$, $j, F, \mathscr{Y}, f, g$ can be defined **directly** in the pure NA world.

$$\begin{tikzcd} X \arrow[d, "\pi"] & X_0 \arrow[l, hook'] \arrow[d, "\pi_0"'] \\ B & B_0 \arrow[l, hook'] \end{tikzcd}$$

Family Floer

T-duality

$$\begin{tikzcd} X_0^\vee \arrow[r, hook, "g"] \arrow[d, "\pi_0^\vee"] & \mathscr{Y} \arrow[d, "f"] \\ B_0 \arrow[r, hook] & B \end{tikzcd}$$



Type (I)

follows family Floer theory and T-duality

We put an analytic topology on

$$X_0^\vee \equiv \bigcup_{q\in B_0} H^1(L_q; U_\Lambda)$$

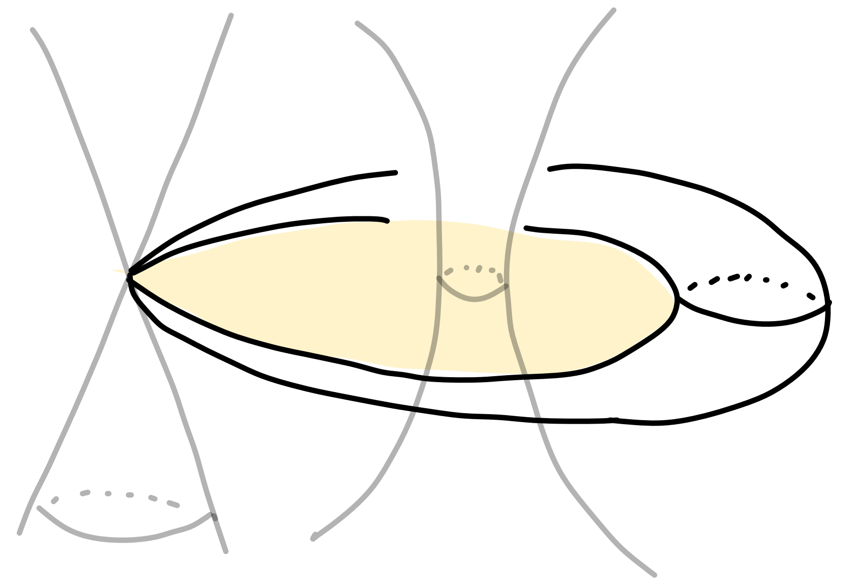
and the wall-crossing of Maslov-0 disks (type (I)) can imply the identification

$$X_0^\vee \equiv T_+ \sqcup T_- / \sim$$

$$\begin{tikzcd}
X_0^\vee \arrow[r, "g"] \arrow[d, "\pi_0^\vee"] & \mathcal{Y} \arrow[d, "f"] \\
B_0 \arrow[r] & B
\end{tikzcd}$$

follows Gross-Hacking-Keel-Siebert's principle based on the identification

agrees with many previous results like Auroux, Abouzaid-Auroux-Katzarkov, Abouzaid-Sylvan, Gammage, Gross-Siebert, etc.



Type (II)

follows Kontsevich-Soibelman further uses type-II disks for singular fibers

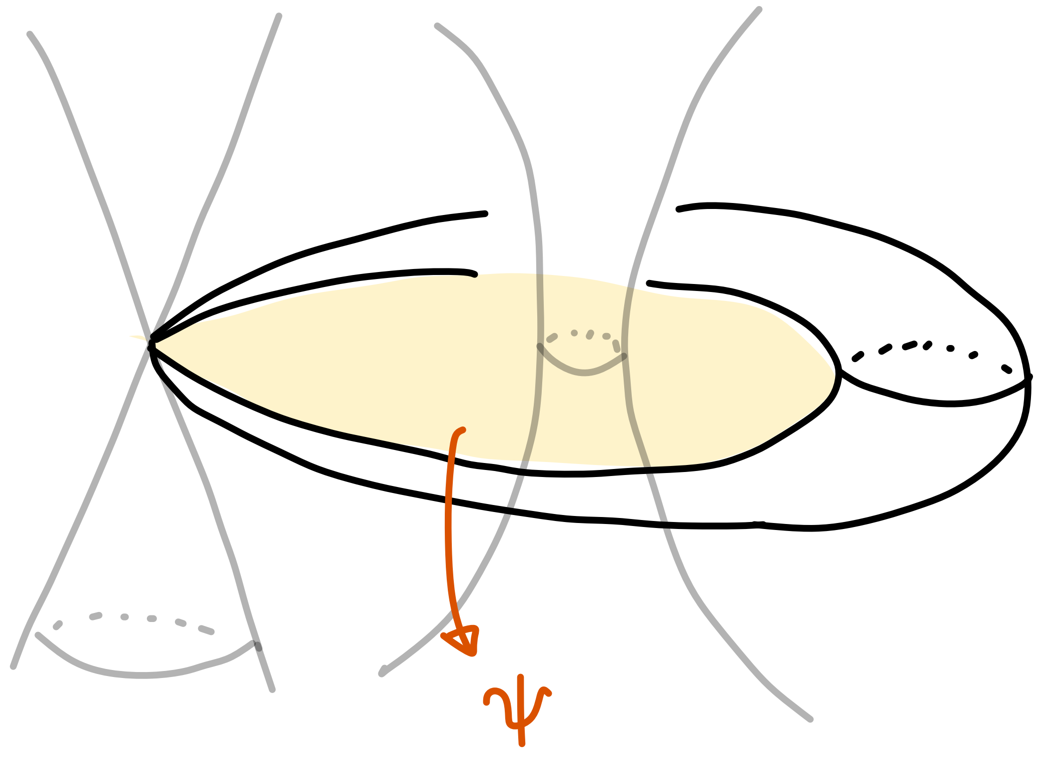
- Given the above, our very explicit T-duality picture is compatible with so many previous mirror symmetry results. Thus, it is very reasonable to believe the analytic fibers of f over $\Delta = B \backslash B_0$ should be the correct **dual singular fibers**
- Meanwhile, by the original family Floer picture, we should expect that $f^{-1}(q)$ is the Maurer-Cartan set $\mathscr{MC}(L_q)$ for $q \in \Delta$
- But unfortunately, the two approaches only have some partial agreements.
- The Maurer-Cartan set is only a strict subset of the corresponding dual singular f-fiber.

Dual singular fiber is not a Maurer-Cartan set !

For simplicity, we assume $n = 2$. Then $B = \mathbb{R}^2$, $\Delta = \{0\}$. Recall $X = \mathbb{C}^2 \backslash \{z_1 z_2 = 1\}$, $Y = \{x_0 x_1 = 1 + y \text{ in } \Lambda^2 \times \Lambda^*\}$

We have a pinched sphere Lagrangian L_0 as the fiber over the singular point 0

Then, the dual singular fiber is (recall that f has an explicit formula)



$$S := f^{-1}(0) = \{(x_0, x_1, y) \in Y \mid \mathsf{v}(x_0) \geq -\psi(0)\ ,\ \mathsf{v}(x_1) \geq \psi(0)\ ,\ \mathsf{v}(y) = 0\}$$

$$\cong \{(x_0, x_1, y) \in Y \mid \mathsf{v}(x_0) \geq 0\ ,\ \mathsf{v}(x_1) \geq 0\ ,\ \mathsf{v}(y) = 0\} \qquad x_0 \mapsto T^{-\psi(0)} x_0\ ,\ x_1 \mapsto T^{\psi(0)} x_1$$

1) If $1 + y \in \Lambda_+$ then $\mathsf{v}(x_0) + \mathsf{v}(x_1) = \mathsf{v}(1 + y) > 0$, so $(x_0, x_1) \in \Lambda_0 \times \Lambda_+ \cup \Lambda_+ \times \Lambda_0$ vice versa.

2) If $1 + y \notin \Lambda_+$ then $\mathsf{v}(x_0) = \mathsf{v}(x_1) = 0$, so (x_0, x_1) is a pair in U_Λ^2 such that $\bar{x}_0 \bar{x}_1 - 1 \neq 0$

In other words, we conclude that

$$S = S_1 \sqcup S_2$$

where

$$S_1 = \Lambda_0 \times \Lambda_+ \cup \Lambda_+ \times \Lambda_0$$

$$S_2 = \{(x_0, x_1) \in U_\Lambda^2 \mid \bar{x}_0 \bar{x}_1 \neq 1\} \cong U_\Lambda \times \left(\mathbb{C}^* \backslash \{-1\} \oplus \Lambda_+\right)$$

On the other hand, Hong, Kim, and Lau have proved that the Maurer-Cartan set for the singular Lagrangian L_0 is ***exactly*** *given by*

$$\mathscr{MC}(L_0) \cong S_1 \quad \subsetneq S$$

- Therefore, $f^{-1}(0) \supsetneq \mathscr{MC}(L_0)$
- There are extra points in S_2 beyond the scope of the *conventional* MC picture.
- One possibility is we need additional 'deformation data' of MC sets of singular Lagrangians.
- The NA analytic topology more or less enforces us to handle singular Lagrangian fibers differently.

Further evidence: a folklore conjecture

Conjecture: (Kontsevich, Seidel, Auroux, …)
The critical values of the mirror Landau-Ginzburg superpotential on $X^\vee$ are the eigenvalues of the quantum multiplication by the first Chern class on X.

- Recall that $X = \mathbb{C}^n \backslash \{z_1 \cdots z_n = 1\}$ and $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$
- The Gross's Lagrangian fibration π can be placed in not only X but also possibly **a larger ambient manifold** $\bar{X}$
- Often, we can check **the Maslov-0 disks keep the same.** Then, we will have the same **analytic topology** for $(X_0^\vee, \pi_0^\vee)$
  Besides, we can also use the same $g : X_0^\vee \to \mathscr{Y}_0$ and $f : \mathscr{Y} \to B$ as before.
- On the other hand, there are no Maslov-2 disks in X, but **there will be new Maslov-2 disks** in $\bar{X}$.
  It gives a global superpotential W_0 on the analytic open domain $X_0^\vee \cong \mathscr{Y}_0$ (using the embedding g).
  Moreover, W_0 is polynomial for our example.
- By our definition of 'SYZ mirror', the analytic domain $\mathscr{Y}_0$ is **Zariski dense** in the algebraic Λ-variety Y.
  Hence, it can be extended on the whole algebraic variety Y, denoted by W.
  (In general, it depends on the Kähler form ω)
- Choosing various ambient space $\bar{X}$ will produce various different Landau-Ginzburg superpotential W on Y

Further evidence: a folklore conjecture

ambient space

$\overline{X} = \mathbb{CP}^n$ Let $H \in \pi_2(\overline{X})$ be the class of a complex line.

LG superpotential

$W = x_1 + \dfrac{T^{E(H)}\ x_0^n}{y_1 \cdots y_{n-1}}$ defined on $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$, where $E(H) = \frac{1}{2\pi}\omega \cap H$

Critical points

There are $n+1$ critical points in a dual analytic f-fiber over $(0,\ldots,0,\ a_\omega) \in B \equiv \mathbb{R}^n$ where $a_\omega \in \mathbb{R}$ depends on the Kähler form ω. Explicitly, these critical points of W are as follows

$$\begin{cases} x_0 = T^{-\frac{E(\mathcal{H})}{n+1}} e^{-\frac{2\pi i s}{n+1}} \\ x_1 = nT^{\frac{E(\mathcal{H})}{n+1}} e^{\frac{2\pi i s}{n+1}} \\ y_1 = \cdots = y_{n-1} = 1 \end{cases} \qquad s \in \{0, 1, \ldots, n\}$$

Remark: There may be many other examples by thinking

(1) other toric CY variety than $\mathbb{C}^n$

(2) other compactification $\overline{X}$

Critical values

$(n+1)T^{\frac{\omega(H)}{n+1}} e^{\frac{2\pi i}{n+1}s}$ for $s \in \{0,1,\ldots,n\}$ One can check the folklore conjecture holds

Further evidence: a folklore conjecture

ambient space	$\overline{X} = \mathbb{CP}^m \times \mathbb{CP}^{n-m}$ for $0 < m < n$ This is also a compactification of $\mathbb{C}^n$ Let $H_1, H_2 \in \pi_2(\overline{X})$ be the classes of a complex line in $\mathbb{CP}^m \times pt$ and in $pt \times \mathbb{CP}^{n-m}$
LG superpotential	1) $W = x_1 + \dfrac{T^{E(H_1)}\ x_0^m}{y_1 \cdots y_m} + \dfrac{T^{E(H_2)}\ x_0^{n-m}}{y_{m+1} \cdots y_{n-1}}$ defined on the same $Y = \{x_0 x_1 = 1 + y_1 + \cdots + y_{n-1}\}$ 2) $W = x_1 + \dfrac{T^{E(H_1)}\ x_0}{y_1} + T^{E(H_2)} x_0$ defined on $Y = \{x_0 x_1 = 1 + y_1\}$, (for $n = 2,\ m = 1$)
Critical points	We have four critical points $\begin{cases} x_0 = \pm T^{\frac{-E(\mathcal{H}_2)}{2}} \\ x_1 = \pm T^{\frac{E(\mathcal{H}_1)}{2}} \pm T^{\frac{E(\mathcal{H}_2)}{2}} \\ y_1 = \pm T^{\frac{E(\mathcal{H}_1) - E(\mathcal{H}_2)}{2}} \end{cases}$ in the fiber of f over $\hat{q} = \left(\dfrac{E(H_1) - E(H_2)}{2}, a_\omega \right) \in B = \mathbb{R}^2$ **Remark**: It may happen that for some Kähler form ω, the number $a_\omega = 0$; If $E(H_1) = E(H_2)$, the $\hat{q}$ is a singular point; we don't know how to prove the folklore conjecture. If $E(H_1) \neq E(H_2)$, the $\hat{q}$ lies on the wall; the conventional proof fails for the Maslov-0 disks, but we can still prove it in my other paper. In general, we don't know if the critical points always avoid the walls or singular locus. It seems the **walls** might be dispersed in an *open subset in* B (e.g. blowup of $\mathbb{C}^n$ along a hyper)
Critical values	$(m+1) T^{\frac{E(H_1)}{m+1}} e^{\frac{2\pi i}{m+1} r} + (n-m+1) T^{\frac{E(H_2)}{n-m+1}} e^{\frac{2\pi i}{n-m+1} s}$ for $r \in \{0, 1, \ldots, m\}$ and $s \in \{0, 1, \ldots, n-m\}$ One can also check the folklore conjecture.

Generalizations

We can repeat the proof almost verbatim for more general examples.

◉ Let $\mathscr{X}_P$ (e.g. $\mathbb{C}^n$) be a toric Calabi-Yau manifold equipped with a toric Kähler form ω corresponding to the (unbounded) moment polytope

$P: \qquad \langle m, v_i \rangle + \lambda_i \geq 0 \quad$ where $m \in M_{\mathbb{R}} \cong \mathbb{R}^n$ and v_i's are the rays in the fan

We may assume $v_1, \ldots, v_n$ form a basis of $N = M^*$. Let $v_{n+a} = k_{a1}v_1 + \cdots + k_{an}v_n$ be the remaining rays. The Calabi-Yau condition means there is $m_0 \in M$ such that $\langle m_0, v_i \rangle = 1$; in particular, $k_{a1} + \cdots + k_{an} = 1$

Note that $v_s - v_n$ $(1 \leq s < n)$ form a basis in the sub-lattice $\langle m_0, \cdot \rangle = 0$. We define

$$X_P = \mathscr{X}_P \backslash (z^{m_0} = 1)$$

It admits a Gross's special Lagrangian fibration π as before.

◉ On the other side, we define a Laurent polynomial

$$h(y_1, \ldots, y_{n-1}) = \sum_{s=1}^{n-1} T^{\lambda_s}(1+\delta_s)y_s + T^{\lambda_n}y_n(1+\delta_n) + \sum_a T^{\lambda_{n+a}}(1+\delta_{n+a})\prod_{s=1}^{n-1} y_s^{k_{as}}$$

The singular locus Δ of π is precisely decided by the tropicalization h_{trop} of h. Here each $\delta_i \in \Lambda_+$ is given by the counts of stable disks with sphere bubbles. Sometimes they are not zero; the valuation $\mathsf{v}(\delta_i)$ is the smallest area of sphere bubble. Finally, we define

$$Y_h = \{(x, y) \in \Lambda^2 \times (\Lambda^*)^{n-1} \mid x_0 x_1 = h(y)\}$$

Theorem: Y_h is SYZ mirror to X_P

Example: Take $h(y) = 1 + y_1 + \cdots + y_{n-1}$, and the corresponding tropical polynomial is

$$h_{\text{trop}} = \min\{0, q_1, \ldots, q_{n-1}\}$$

which somehow plays the leading role. For instance

i. describes the walls on the A side
ii. appears in the formula of f i.e. the pair (j, F) on the B side
iii. gives the singular locus $\Delta = B \backslash B_0$

"SYZ converse": Given h_{trop} and h, we conversely have $P' := \{(\bar{q}, q_n) \mid q_n + h_{\text{trop}}(\bar{q}) \geq 0\} \cong P$

This picture will be lost if we only work over $\mathbb{C}$.

Example: $h(y) = y_1 + T^{-1}y_2 + T^{3.14} + T^2 y_1^2 + y_1 y_2 + T^2 y_2^2$

Then, we can recover P:

Have infinite such examples.

$v_1 = (1,0,0)$	$\lambda_1 = 0$
$v_2 = (0,1,0)$	$\lambda_2 = -1$
$v_3 = (0,0,1)$	$\lambda_3 = 3.14$
$v_4 = (2,0,-1)$	$\lambda_4 = 2$
$v_5 = (1,1,-1)$	$\lambda_5 = 0$
$v_6 = (0,2,-1)$	$\lambda_6 = 2$

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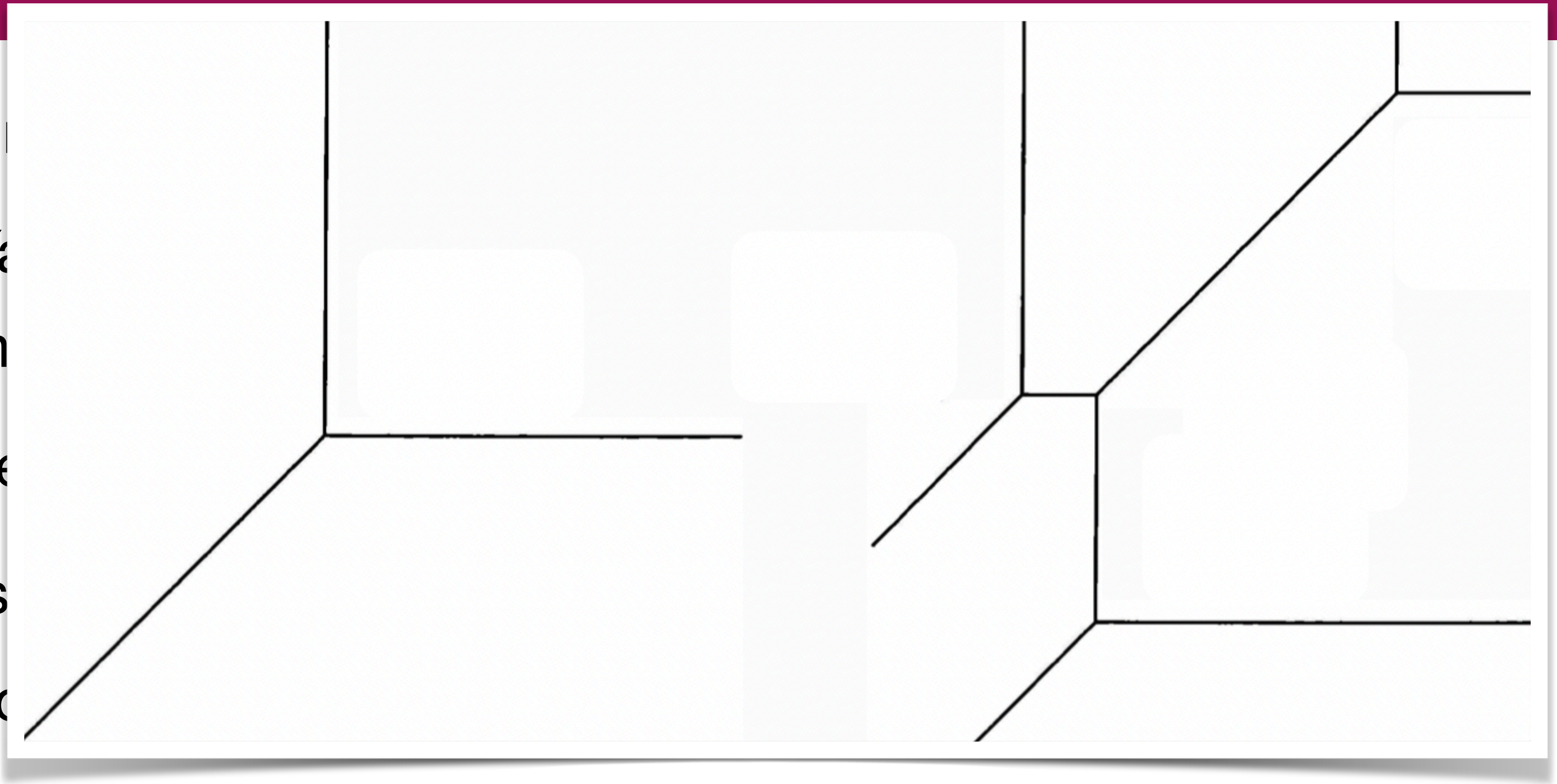

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$v_4 = (2,0,-1) \qquad \lambda_4 = 2$
$v_5 = (1,1,-1) \qquad \lambda_5 = 0$
$v_6 = (0,2,-1) \qquad \lambda_6 = 2$

Thanks for your attention !