Family Floer SYZ conjecture and examples

§§ Review

§ Symplectic & non-archimedean integrable system.

① Arnold-Liouville's theorem : (Symplectic).

- any Lagrangian fibration $\pi : X^{\omega} \longrightarrow B$
  admits action-angle coordinates over a nbhd of a smooth point $q \in B_{smooth}$

- So, $q \in B$ is smooth if $\exists\ U \ni q$ s.t.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\cong} & \mathrm{Log}^{-1}(V) \\ \downarrow & & \downarrow \\ U & \xrightarrow{\chi} & V \end{array}$$

where

$*$ $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$. $z_k \mapsto \log|z_k|$. $\rightsquigarrow r_k$.

$$(\mathbb{C}^*)^n = \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n$$

$z_k = e^{r_k + i\theta_k}$, $\quad \underset{\text{action}}{r_k \in \mathbb{R}}$, $\quad \underset{\text{angle}}{\theta_k \in \mathbb{R}/2\pi\mathbb{Z}}$.

* ω is identified with $\sum_k dr_k \wedge d\theta_k$

* $\chi : U \longrightarrow V$ is an <u>integral affine chart</u>

<u>Recall</u> • an integral affine structure refers to an atlas of coordinate charts so that the transition maps are $x \mapsto Ax + b$

for $A \in GL(n, \mathbb{Z})$, $b \in \mathbb{R}^n$.

• An integral affine structure with singularities.

$$B = \underset{\text{smooth}}{B_0} \cup \underset{\text{singular locus.}}{\Delta}$$

② <u>Kontsevich–Soibelman</u> (Non-archimedean).

$f : \mathcal{Y} \longrightarrow B$ a (tropically) continuous map w.r.t Berkovich topology in $\mathcal{Y}$ and Euclidean topology in B.

In analogy, $q \in B$ is called smooth if $\exists\, U \ni q$ s.t.

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\cong", "\text{as Berkovich analytic spaces}"'] \arrow[d, "f"] & \mathrm{trop}^{-1}(V) \arrow[d, "\mathrm{trop}"] \\ U \arrow[r, "\chi"] & V \end{tikzcd}$$

where

* $\mathrm{trop} : \left(\mathrm{Spec}\,\Lambda[x_1^{\pm} \dots x_n^{\pm}]\right)^{an} \longrightarrow \mathbb{R}^n$.

$\qquad\quad \mathrm{|||}$ $\leftarrow$ set of closed pts.

$\qquad (\Lambda^*)^n$

$z_k \longmapsto -\log|z_k| = v(z_k)$

$$\Lambda = \mathbb{C}((T^{\mathbb{R}})) = \left\{ x = \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \nearrow \infty \right\}.$$

is the Novikov field with NA $\begin{cases} \text{valuation} & v(x) = \lambda_0 \ (a_0 \neq 0) \\ \text{norm} & |x| = e^{-v(x)}. \end{cases}$

NA triangle ineq

$v(x+y) \geq \min\{v(x), v(y)\}.$

$|x+y| \leq \max\{|x|, |y|\}.$

* trop is called the tropicalization map NA analog.
  and serves as the local model of affinoid torus fib

  Def every point in the base is smooth.

* $\chi : U \to V$
  gives rise to an integral affine chart.
  (need to use NA topology).

§8 Family Floer T-duality construction

- There are two natural methods for equipping a base manifold with an integral affine structure.

  (SG or NA)

- Let's begin with a Lagrangian fibration with singularities.

$$\pi : X \longrightarrow B. \qquad \begin{cases} B_0 \subseteq B & \text{sm locus} \\ \Delta \subseteq B & \text{sing locus.} \end{cases}$$

- We can cover B_0 by small integral affine charts.

$$\begin{array}{ccc} B_0 & & \mathbb{R}^n \\ \cup\!\mid & & \cup\!\mid \\ \chi_i : U_i & \longrightarrow & V_i \end{array}$$

- Clearly, $\pi^{-1}(U_i) \cong \mathrm{Log}^{-1}(V_i) \subseteq (\mathbb{C}^*)^n$

$$\begin{array}{ccc} \pi^{-1}(U_i) & & \mathrm{Log}^{-1}(V_i) \\ \vdots & & \vdots \\ U_i & \dashrightarrow & V_i \end{array}$$

can glue to $\pi_0 = \pi|_{B_0}$. a Lag torus fibration

- **Question** Let's artificially consider the analytic domains $\mathrm{trop}^{-1}(V_i) \subseteq (\Lambda^*)^n$

Can we glue them to a NA affinoid torus fibration?

- **My thesis** There is a unique canonical way to assemble the collection $\{\mathrm{trop}^{-1}(V_i)\}$ by using the data of Maslov-0 holo disks bounded by π_0-fibers.

(under some conditions)

sufficient conditions for simplicity

① special Lagrangian fibration (graded, zero Maslov class)

stable Lag?

② involution-invariant ($\exists\, \phi: (X,\omega) \to (X,-\omega)$. ϕ preserve π_0-fibers).

(e.g. satisfied by

the local model $\mathrm{Log}: (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$, $z_k \mapsto \log|z_k|$.

$\phi: z_k \mapsto \bar{z}_k$. global?

Theorem (My thesis):

$\exists$ a non-archimedean analytic space $X_0^\vee$ over Λ.

a global analytic function $W_0^\vee$

a dual affinoid torus fibration

$$\pi_0^\vee : X_0^\vee \longrightarrow B_0$$

such that

(a) unique up to isomorphism

(b) The integral affine str induced by $\pi_0^\vee$ coincides with the one induced by π_0

(c) $$X_0^\vee \overset{\text{set}}{=\!=} \bigcup_{q \in B_0} H^1(L_q ; U_\Lambda)$$

unit circle in Λ

like

$U(1)$ in $\mathbb{C}$

$$\pi_0^\vee \overset{\text{set}}{=\!=} \text{“} H^1(L_q ; U_\Lambda) \longmapsto q \text{''}$$

Local picture

- Denote the natural pairing

$$\pi_1(L_q) \times H^1(L_q; U_\Lambda) \longrightarrow U_\Lambda$$

by $(\alpha, y) \longmapsto y^\alpha$

- Given a "pointed" integral affine chart

$$\chi : (U, q_0) \longrightarrow (V, c_0). \qquad \chi(q_0) = c_0$$

we have a NA analytic chart

$$\begin{array}{ccc}
(\pi_0^\vee)^{-1}(U) & \xrightarrow{\ \cong\ } & \mathrm{trop}^{-1}(V - c) \\
\| \wr & & \\
\bigcup_{q \in U} H^1(L_q; U_\Lambda) & & \\
\downarrow \pi_0^\vee & & \downarrow \\
U & \xrightarrow{\ \chi = (\chi_1, \dots, \chi_n)\ } & V
\end{array}$$

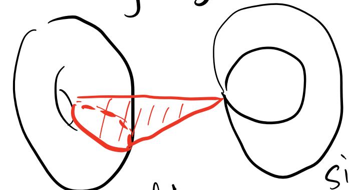
by identifying

$$\underset{H^1(L_q; U_\Lambda)}{\overset{\cap}{y}} \longleftrightarrow \left(T^{\chi_1} y^{e_1}, \dots, T^{\chi_n} y^{e_n} \right)$$

where $(e_1 \dots e_n)$ basis of $\pi_1(L_q)$
compatible with $(\chi_1, \dots, \chi_n)$

- My thesis tells you we can always glue them canonically.
  i.e. $\pi_0^\vee : X_0^\vee \to B_0$

- It uses data of not only π_0 but also π.
  because holomorphic disk are essentially global



Singular extension

Question Since we begin with π,
it is natural to ask if we can extend $\pi_0^\vee : X_0^\vee \to B_0$
to some "$\pi^\vee : X^\vee \to B$"?

Principle There should be no essential difficulty to do so.

Reasons: ① The $\pi_0^\vee$ already captures a substantial amount of information about the singular π-fibers.

② NA analytic structure of $(X_0^\vee, \pi_0^\vee)$ is very "rigid"
(compare the rigidity of holomorphic functions & cx mfds)
The freedom for the singular extension is very limited.

③ We have many explicit & elementary examples to justify this principle.

Family Floer SYZ conjecture

Let X be Calabi-Yau with a holo volume form Ω.

$\exists$ a Lagrangian fibration $\pi : X \longrightarrow B$ (graded/special w.r.t Ω)

$\exists$ a tropically continuous map $f : \mathcal{Y} \longrightarrow B$
from a Berkovich analytic space $\mathcal{Y}$ over $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$.

such that

nontrivial
(i) π and f have the same singular locus skeleton Δ in B.

(ii) $\pi_0 = \pi|B_0$ and $f_0 = f|B_0$ induce the same integral affine str on $B_0 = B \setminus \Delta$.

(iii) f_0 is isom. to the canonical dual affinoid torus fibration $\pi_0^\vee$

$$\begin{array}{ccccccc} X & \longleftarrow & X_0 & \text{- - - - -} & \mathcal{Y}_0 & \longrightarrow & \mathcal{Y} \\ \pi \downarrow & & \pi_0 \downarrow & \begin{array}{c}\text{T-duality} \\ \text{with} \\ \text{quantum} \\ \text{correction.}\end{array} & \downarrow f_0 \cong \pi_0^\vee & & \downarrow \\ B & \longleftarrow & B_0 & \text{- - - - -} & B_0 & \longrightarrow & B \end{array}$$

Rmks

- So far, we have established several examples.

§§ Example : conifold

$$Z = \{(u_1, v_1, u_2, v_2) \in \mathbb{K}^4 \mid u_1 v_1 = u_2 v_2\}.$$

$\boxed{\text{A-side}}$ over $\mathbb{C}$. conifold smoothing

$$X' = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 - c_1 = u_2 v_2 - c_2\}$$

for $c_1 > c_2 > 0$

Remove the divisor

$$\mathcal{D} = \{u_1 v_1 - c_1 = u_2 v_2 - c_2 = 0\}.$$

Set $X = X' \setminus \mathcal{D} = \left\{ (u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^* \;\middle|\; \begin{array}{l} u_1 v_1 - c_1 = z \\ u_2 v_2 - c_2 = z \end{array} \right\}$

Special Lagrangian fibration on X.

$$\pi : (u_1, v_1, u_2, v_2, z) \mapsto \left(\frac{|u_1|^2 - |v_1|^2}{2}, \frac{|u_2|^2 - |v_2|^2}{2}, \log |z| \right)$$

Two Hamiltonian S^1-actions

$$(u_j, v_j) \mapsto (e^{it} u_j, e^{-it} v_j).$$

Fixed points $\hat{C}_1 = \{u_1 = v_1 = 0, \; z = -c_1\}$

$$\hat{C}_2 = \{u_2 = v_2 = 0, \; z = -c_2\}$$

singular locus

$$\Delta = \Delta_1 \cup \Delta_2 \quad \begin{cases} \Delta_1 = 0 \times \mathbb{R} \times \log c_1 = \pi(\hat{C}_1) \\ \Delta_2 = \mathbb{R} \times 0 \times \log c_2 = \pi(\hat{C}_2). \end{cases}$$

$\boxed{\text{B-side}}$ over the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$

conifold resolution

$$Y' = \mathcal{O}_{\mathbb{P}_\Lambda}(-1) \oplus \mathcal{O}_{\mathbb{P}_\Lambda}(-1).$$

Remove a divisor, we get an algebraic variety.

$$Y = \left\{ (x_1, x_2, z, y_1, y_2) \in \Lambda^2 \times \mathbb{P}_\Lambda \times (\Lambda^*)^2 \;\middle|\; \begin{array}{l} x_1 z = 1 + y_1 \\ x_2 = (1+y_2) z \end{array} \right\}$$

Define a Berkovich analytic space

$$\mathcal{Y} = \{ |x_2| < 1 \text{ in } Y^{an} \}.$$

(We use Berkovich NA topology finer than Zariski) the action coordinates locally over B_0. Define a non-archimedean analytic space $\mathscr{Y} = \{|x_2| < 1\}$ in the analytification Y^{an} of Y. Define a continuous embedding $j : \mathbb{R}^3 \to \mathbb{R}^5$ sending $q = (q_1, q_2, q_3)$ to

$$(\theta_1(q), \theta_2(q), \vartheta(q), q_1, q_2)$$

where

$$\begin{aligned}
\theta_1(q) &= \min\{-\psi(q), -\psi(q_1, q_2, \log c_1)\} + \min\{0, q_1\} + \min\{0, q_2\} \\
\theta_2(q) &= \min\{\quad \psi(q), \quad \psi(q_1, q_2, \log c_2)\} \\
\vartheta(q) &= \mathrm{median}\{\psi(q), \quad \psi(q_1, q_2, \log c_1), \quad \psi(q_1, q_2, \log c_2)\}
\end{aligned}$$

Define a tropically continuous map $F : Y^{an} \to \mathbb{R}^5$ by

$$F(x_1, x_2, z, y_1, y_2) = (F_1, F_2, G, \mathsf{v}(y_1), \mathsf{v}(y_2))$$

where

$$\begin{aligned}
F_1 &= \min\{\mathsf{v}(x_1), -\psi(\mathsf{v}(y_1), \mathsf{v}(y_2), \log c_1) + \min\{0, \mathsf{v}(y_1)\} + \min\{0, \mathsf{v}(y_2)\}\} \\
F_2 &= \min\{\mathsf{v}(x_2), \quad \psi(\mathsf{v}(y_1), \mathsf{v}(y_2), \log c_2)\} \\
G &= \mathrm{median}\{\mathsf{v}(z) + \min\{0, \mathsf{v}(y_2)\}, \quad \psi(\mathsf{v}(y_1), \mathsf{v}(y_2), \log c_1), \quad \psi(\mathsf{v}(y_1), \mathsf{v}(y_2), \log c_2)\}
\end{aligned}$$

By tedious but routine computations, we can check $j(\mathbb{R}^3) = F(\mathscr{Y})$. So, we can define (cf. [20, §8])

$$f = j^{-1} \circ F : \mathscr{Y} \to \mathbb{R}^3 \tag{5}$$

<u>looks ridiculous, but:</u>

- There are underlying geometric meanings from family Floer theory.
- The NA geometry is really "<u>inherent & intrinsic</u>" from the symplectic / Kähler geometry. (e.g. Gromov's compactness)
- The function ψ reflects the reduced Kähler geometry for T^2-quotient. (The ψ is given by certain symplectic areas the various reduced spaces)
- The various <u>min & median</u> involve certain comparisions of these symplectic areas, which finaly constitutes the NA structure.
- In a word, we cay say (oversimplified)

  Symplectic area "is" the mirror NA analytic topology.

An elementary experiment (extra evidence)

Let's forget NA topology and only think of Zariski topology

The algebraic variety $Y = \left\{ \begin{array}{l} x_1 z = 1 + y_1 \\ x_2 = (1+y_2) z \end{array} \right\}$. $x_1, x_2 \in \mathbb{K}$. $y_1, y_2 \in \mathbb{K}^*$ $z \in \mathbb{P}_{\mathbb{K}}$.

has three algebraic torus charts (Zariski open dense).

$T_1 = \{x_1 \neq 0\}$: $(x_1, y_1, y_2) \in (\mathbb{K}^*)^3$: $z = \frac{1+y_1}{x_1}$

$T_2 = \{x_2 \neq 0\}$ . . .

$T_3 = \{0 \neq z \neq \infty\}$. $(z, y_1, y_2) \in (\mathbb{K}^*)^3$: $x_1 = \frac{1+y_1}{z}$, $x_2 = (1+y_2) z$

Codimension-2 missing points

$$Y \setminus T_1 \cup T_2 \cup T_3 = C_1 \sqcup C_2$$

where $\left\{ \begin{array}{l} C_1 = \{x_1 = x_2 = 0,\ y = -1,\ z = 0\} \\ C_2 = \{x_1 = x_2 = 0,\ y = -1,\ z = \infty\} \end{array} \right.$

Now, let's go back to the formula of $f = j^{-1} \circ F$.

(very unmotivated & crazy)

sing locus.

But, magically,

$$(*) \quad \begin{cases} f(C_1) = \Delta_1 \\ f(C_2) = \Delta_2 \end{cases} \Rightarrow f(C_1 \cup C_2) = \Delta$$

also A-side

Since f has an elementary formula,

checking $(*)$ is really an easy exercise (even for students).

§§ Example A_n - singularity

$$uv = z^{n+1}.$$

<u>A-side</u> (A_n-smoothing) $\quad uv = \prod_{k=0}^{n} (z - a_k).$

<u>B-side</u> (A_n-resolution)

a toric surface associated to the fan

- We can still prove Family Floer SYZ conjecture (Work in progress).
- The special Lagrangian fibration takes the form

$$\pi(u, v, z) = \left(\frac{|u|^2 - |v|^2}{2}, \; |z - z_0| \right) \quad \text{for some } z_0.$$

By deforming z_0, or by deforming $a_0, a_1 \cdots a_n$
we expect a deformation of NA analytic structure

(e.g. The mirror torus fibration (not mirror space) may be different
when $|a_0| = |a_1| = \cdots = |a_n|$ or $|a_0| < |a_1| < \cdots < |a_n|$.)

- May discover relation to Braid group action
  (Khovanov, Seidel, Thomas)
  with concrete geometric meanings.

§§ Potential future projects

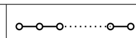
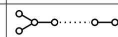
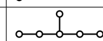
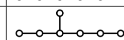
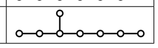
- Open GW inv and SYZ under conifold transitions
  (Siu-Cheong Lau 2013)

  * smoothing of more general toric Gorenstein singularities.

  * Mirror spaces are expected to be the same
    But, mirror fibrations are a totally different and new story. (more challenging)
    This may discover new structural results.

  * Relation to Minkowski decomposition?

- Other types of ADE singularities.

$A_{n\geq 1}$	$xy + z^{n+1}$	
$D_{n\geq 4}$	$x^2 + y(z^2 + y^{n-2})$	
E_6	$x^2 + y^3 + z^4$	
E_7	$x^2 + y(y^2 + z^3)$	
E_8	$x^2 + y^3 + z^5$	

Issue. Can we find Lagrangian fibration on their smoothings?

(Advantage of using family Floer: zero Maslov class is enough).

- SYZ mirror Symmetry for Hypertoric Varieties (Lau-Zheng).

(Advantage of family Floer: we may deal with codim-1 singular locus)

Two main steps (General Principle)

① "Topological wallcrossing"

completely understand the local system

$$\bigsqcup_{q \in B_0} \pi_2(X, L_q).$$

and how they "degenerate" as $q \to \Delta = B \setminus B_0$.

② "Local superpotential computations"

$\downarrow$

in each chamber.

It is fine if we cannot explicitly do so.

$$\{\text{partial compactification}\} \longrightarrow \{\text{global analytic functions}\}$$

sufficient $\rightsquigarrow W_1, W_2, \cdots, W_n.$

- We don't have to explicitly find W_k's.
  We just need to understand how W_k's are related to each other.
  This is somehow similar to Kodaira embedding

$$X_0^\vee \xrightarrow{(g_1, g_2, \cdots g_N)} \Lambda^N$$

for $N \gg 1$.

* g_k are some combination of global analytic functions so that $\mathrm{Zero}(g_k)$ are distinct.

③ All current examples of [illegible]

(3) All current examples of Family Floer SYZ conjecture essentially follow this general principle.