Non-archimedean analytic continuation of unobstructedness.

Main result $(L_s : s \in S)$ is a smooth family of graded or vanishing Maslov index Lagrangian submanifold in (M, ω). Assume S is connected. Let $s_0 \in S$

Theorem If L_{s_0} is properly unobstructed, then all L_s is properly unobstructed

Remark

monotone $\Longrightarrow$ properly unobstructed $\Longrightarrow$ (a subcase of) usual unobstructed in the literature (FOOO & Kontsevich).
$\not\Longleftarrow$

(will explain soon)
only involve Maslov-0 disks

Maslov index always > 0 $\Longrightarrow$ properly unobstructed

Application: ① Rizell - Goodman - Ivrii:

Let (X, ω) be either of $(\mathbb{R}^4, \omega_{std})$, $(\mathbb{CP}^2, \omega_{FS})$, $(S^2 \times S^2, \omega_1 \oplus \omega_1)$

Any two Lagrangian tori inside (X, ω) are Lagrangian isotopic.

By our result

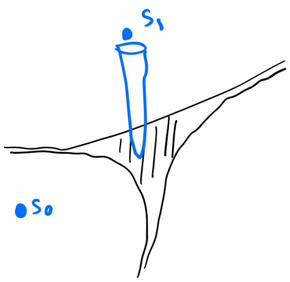
Corollary Any Lagrangian tori in (X, ω) is (properly) unobstructed

Proof Pick a monotone Lag tori, ...

② Abouzaid - Auroux - Katzarkov

Lagrangian torus fibration on blowups of toric mfd. "negative vertex"

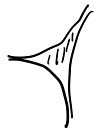
$\pi : X \longrightarrow B$ e.g. $X = \{uv = 1 + w_1 + w_2\}$, $B = \mathbb{R}^3$

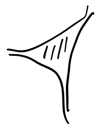


<u>singular locus</u> $\Delta \subseteq B$

<u>Wall region</u>

$$\Omega = \{ x \in B_0 \mid L_x \text{ bounds Maslov-0 disk} \}$$

then $\Omega \cong$  $\times \mathbb{R} \setminus \{0\}$

$\Delta \cong$  $\times \{0\}$.

<u>Corollary</u>

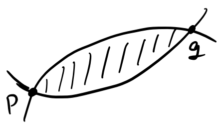
unobs of L_{s_0} (trivial)

$\Downarrow$

unobs of L_{s_1} (nontrivial) as L_{s_1} bounds Maslov-0 disks.

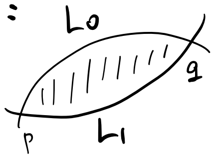
• <u>Lagrangian Floer cohomology</u> (Review): L_0 L_1 transversal

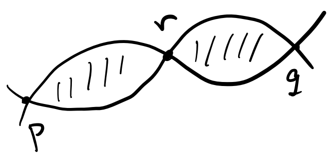
$CF(L_0, L_1)$ "=" $\langle L_0 \cap L_1 \rangle$

$\circlearrowright$ δ counts 

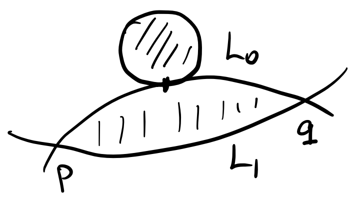
$\delta(q)$ "=" $\#(\cdots)\cdot p$

Hope $HF(L_0, L_1) = H^*(CF(L_0, L_1), \delta)$

"$\delta \cdot \delta = 0$": "∂" 



in general



• <u>Obstruction</u>:

$\delta \circ \delta \neq 0$ in general.

bubbling off disks is a phenomenon of codimension one (not two). → affect Stokes

even a single disk

$\Downarrow$

curved A_∞ algebras

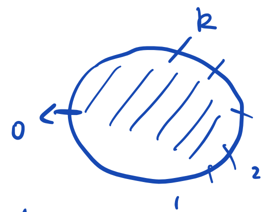
- A_∞ algebra associated to a Lagrangian submanifold

$\forall k \geq 0, \forall \beta \in H_2(X, L)$

* a collection of multilinear maps   or $\boxed{\check{m}_k = \sum T^{\omega(\beta)} \check{m}_{k,\beta}}$

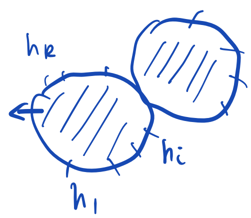
$$\check{m}_{k,\beta} : \Omega^*(L)^{\otimes k} \longrightarrow \Omega^*(L)$$

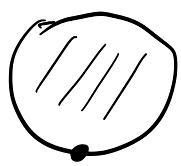
of degree $2 - k - \mu(\beta)$. (Maslov index.) such that



$\mathcal{M}_{k+1,\beta}(J, L)$

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=0}^{k_1} (-1)^* \check{m}_{k_1,\beta_1}\left(h_1 \cdots h_i, \check{m}_{k_2,\beta_2}(\cdots) \cdots h_k\right) = 0$$



* The curvature term $\check{m}_{0,\beta}$ counts  (1 marked pt.)

$\rightsquigarrow$ obstruction of "$\delta \circ \delta = 0$"

$\Rightarrow \delta \circ \delta \neq 0$

~~(Weak)~~

- **Bounding cochain** (FOOO & Kontsevich)

$b = \sum T^{\lambda_i} b_i \in \Omega^{odd}(L) \otimes \Lambda_0$ such that

$$\sum_{k=0}^{\infty} \sum_{\beta} T^{\omega(\beta)} m_{k,\beta}(b \cdots b) = 0 \quad \cancel{(\text{or } c \cdot \mathbb{1})}$$

Defn People call L *unobstructed* if $\exists$ such b

Then, we can "formally" define a deformed A_∞ algebra

$$m_k^b(x_1, \cdots, x_k) = m(b \cdots b\ x_1, b \cdots b, x_2, \cdots\cdots)$$

with the curvature term $m_0^b = 0$.

Question any other approach possible ?

Reasons to rethink

① The existence of b is usually unknown.

(2) making a choice always means loss of information.

* That's why we need to study what happens for a different choice.
* In the case of bounding cochains, we get "gauge equivalence".

$\rightsquigarrow$ Maurer-Cartan set

But, only a set.

FOOO: the solution space of b modulo gauge equivalence is set theoretically invariant up to homotopy equivalence of A_∞ algebras.

$\downarrow$

- lose info
- unpractical for explicit computation.

$\downarrow$

not strong enough in practice

e.g. HH^* as a ring is NOT invariant

pro: work for most general case: even negative index

However

in practice, we often like graded or vanishing Maslov index Lagrangian submanifolds. (Calabi-Yau, mirror symmetry ...)

$\Longrightarrow \mu(\beta) \geq 0$ when $\beta \in H_2(X, L)$ is represented by a holo disk u

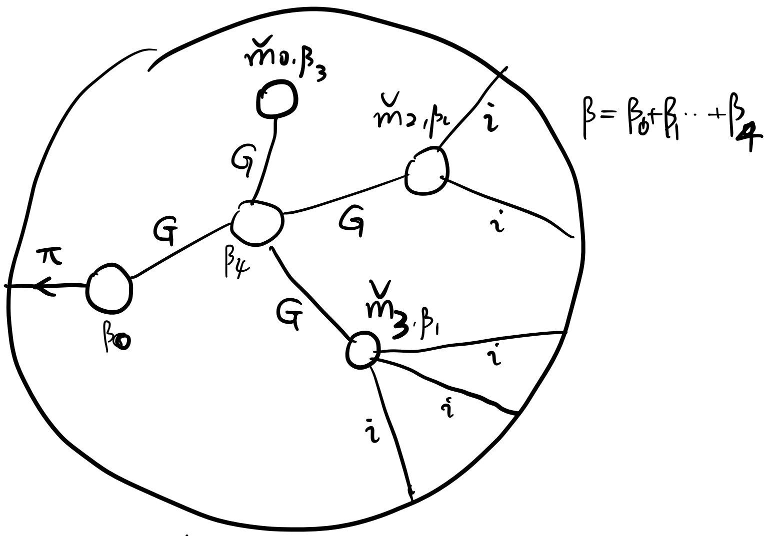
Let's propose a different notion of unobstructedness.

By homological perturbation, the $\check{m}$ on $\underbrace{\Omega^*(L)}_{\dim = \infty}$ induces

$$m_{k,\beta} : H^*(L)^{\otimes k} \longrightarrow \underbrace{H^*(L)}_{\dim < \infty\,!}$$

Intuitively:

$$m_{k,\beta} = \sum$$



"holomorphic pearly trees"

Remark We know $\check{m} \underset{\text{homotp equiv of } A_\infty \text{ alg}}{\simeq} m$. We know $MC(\check{m}) \overset{\text{set}}{\cong} MC(m)$

But, homotopy equivalence (quasi-isomorphism) may be not strong enough

e.g. we know very little about $HH^*(\check{m})$ v.s. $HH^*(m)$

e.g. $\underset{g.J}{m}$ on $H^*(L)$ captures the info of a Riemannian metric, but $\underset{J}{\check{m}}$ on $\Omega^*(L)$ does not.

Recall $\mu(\beta) \geq 0$

$\deg m_{k,\beta} = 2 - k - \mu(\beta)$.

$$\Rightarrow \quad m_{0,\beta} \in H^{2-\mu(\beta)}(L) \quad \begin{cases} H^0(L) & \text{Maslov}-2 \\ H^2(L) & \text{Maslov}-0 \end{cases}$$

Denote the basis of $H^0(L)$, $H^2(L)$ by $\mathbb{1}$, $\Theta_1 \cdots \Theta_\ell$

- We write

$$\sum T^{E(\beta)} \underset{=}{Y}^{\partial\beta} m^{J}_{0,\beta} = W^J \cdot \mathbb{1} + \sum_{i=1}^{\ell} Q_i^J \cdot \Theta_i$$

" formal symbol "

where $W, Q_i \in \Lambda[[H_1(L)]]$.

(like group ring)

- For different choice, we also have m', W', Q_i'

(Key)

- **Theorem A** $\quad Q_i \equiv 0 \ \forall i$ iff $Q_i' \equiv 0 \ \forall i$

necessary

Defn We say L is properly unobstructed if $Q_i \equiv 0$

Prop properly unobstructed $\Longrightarrow$ usual unobstructed

Roughly, if $(y_1 \cdots y_n)$ is a zero of Q_i's then x_i "=" $\log y_i$ gives a usual bounding cochain

Recall:

Main Theorem If L_{s_0} is p. unob, then all L_s are p. unob.

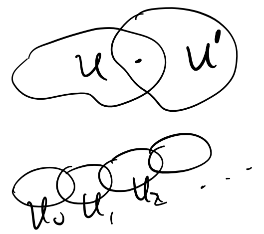
$\{L_s : s \in S\}$

Idea

Consider $A = \{ s \in S \mid L_s \text{ is properly unobstructed} \}$ well-defined due to Theorem A

$$? \begin{cases} \text{non-empty} & \checkmark \\ \text{open} & \text{Fukaya's trick} + \boxed{\text{Theorem A}} \\ \text{closed} & \text{a uniform version of reverse isoperimetric inequality} \end{cases}$$

- Note that the convergence domain $\mathcal{U}$ of Q_i may differ from the convergence domain $\mathcal{U}'$ of Q_i'



- To give some concrete insight,

Prop Let $f = \sum_{\nu \in \mathbb{Z}^n} c_\nu Y^\nu \in \Lambda[[Y_1^{\pm}, \dots, Y_n^{\pm}]]$ $\quad U_\Lambda = \{ x \in \Lambda \mid |x| = 1 \}$

f is identically zero iff f is $\overset{(NA)}{\text{convergence}}$ and vanishing on U_Λ^n

Proof "$\Leftarrow$": $|c_\nu| \to 0$ as $|\nu| \to \infty$ (NA norm). Arguing by contradiction.

may assume $c_{\nu_0} = 1$ and $|c_{\nu_0}| = \max |c_\nu| = 1$

Modulo the ideal of elements with norm < 1, we get

$\bar{f} = \sum \bar{c}_\nu Y^\nu$ and $\bar{c}_\nu = 0$ for $|\nu| \gg 1$.

Hence, $\bar{f}$ is just a Laurent polynomial with $\bar{c}_{\nu_0} = 1$ $\Leftarrow$ $\lightning$

But f vanishing on $U_\Lambda^n \Rightarrow \bar{f}$ vanishing on $(\mathbb{C}^*)^n \Rightarrow \bar{f} = 0$

$\square$

Sketch of proof of Theorem A

By Groman-Solomon's reverse isoperimetric ineq.

$W, Q_i, W', Q_i' \in \Lambda\langle\Delta\rangle \subsetneq \Lambda[[H_1(L)]]$ (with some adic convergence cond.)

an affinoid algebra, say A

Consider obstruction ideals $\mathfrak{a} = (Q_i)$, $\mathfrak{a}' = (Q_i')$. (noetherian) like AG

We aim to find an isomorphism $A/\mathfrak{a}' \overset{\varphi}{\cong} A/\mathfrak{a}$ of affinoid algebra.

Roughly,

the formula of φ is given by a pseudo-isotopy (due to Junwu Tu and Fukaya)

- But, one of the main problems is to find φ^{-1}.

  in the category of affinoid algebras.

  (not just in the category of sets)

- Namely, we want to achieve $\varphi^{-1} \circ \varphi = id$ and $\varphi \circ \varphi^{-1} = id$.

  A key new point is to ensure the divisor axiom is functorial and especially is preserved when taking the homotopy inverse of an A_∞ quasi-isomorphism.

- Moreover, we can talk about the dimension of affinoid algebras.

  The maximal ideal of $A \longleftrightarrow$ a point in some $\mathcal{U} \subseteq (\Lambda^*)^n$

  $$A/\mathfrak{a} \longleftrightarrow \text{Zero locus of } \mathfrak{a} \text{ in } \mathcal{U}.$$

Concluding Remark :

- coefficient in Lagrangian Floer

$\mathbb{C} \rightsquigarrow \Lambda$ Novikov field $\rightsquigarrow$ $\boxed{\text{affinoid algebra over } \Lambda}$

- Motivation from Mirror Symmetry :

We can also define $D^b Coh$ on a non-archimedean analytic space.

Coherent sheaf $\longleftrightarrow$ module over affinoid algebra

- Thus, it's natural to expect $HF^*(L_1, L_2)$ is a module over affinoid algebra.

- An ultimate understanding of mirror symmetry should be like an intrinsic identification

$$\text{"} D\mathcal{F}uk(X) \text{"} \cong D^b Coh(X^\vee)$$

$D\mathcal{F}uk(X)$: over affinoid alg

$X^\vee$: NA analytic space