SYZ Mirror Symmetry for A_n singularity

① A Calabi-Yau manifold admits a special Lagrangian $\pi : X \to B$ with some singular locus $\Delta \subseteq B$.
(Yang Li, Tristan Collins, etc.)

Goal ② Mirror manifolds should be constructed as another torus fibration $\pi^\vee : X^\vee \to B$.
by fiberwise "dualizing" the Lagrangian torus fibers

- Gross's Topological Mirror Symmetry $\Big\langle$ include singular fibers / but only in topological level.

- Except this, the evidence for ② is rare.

Q = More structure for ② ?

§ Toy SYZ model

- $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ Lag. fib.
  $z_k \longmapsto \log |z_k|$

- We "claim" its SYZ dual fibration is

$$\mathrm{trop} : (\Lambda^*)^n \longrightarrow \mathbb{R}^n$$
$$z_k \longmapsto -\log|z_k| \equiv v(z_k).$$

Novikov field
$\Lambda = \mathbb{C}((T^{\mathbb{R}}))$, $\begin{cases} a_0 \neq 0 \\ \lambda_0 < \lambda_1 < \cdots < \lambda_n \nearrow \infty \end{cases}$
$v\left(\sum_{i=0}^{\infty} a_i T^{\lambda_i}\right) = \lambda_0$, $|\cdot| = e^{-\lambda_0}$

<u>Note</u>

- $(\mathbb{C}^*)^n$ is a Kähler manifold
  $(\Lambda^*)^n$ is a Berkovich space.

$$\begin{tikzcd} & \text{over } \mathbb{C} \arrow[r, "|\cdot|_{\mathbb{C}}"] & \text{complex topology} \\ \text{algebraic variety (Zariski)} \arrow[ur, dash] \arrow[dr, dash] & & \\ & \text{over } \Lambda \arrow[r, "|\cdot|_{\Lambda}"] & \text{Berkovich topology} \end{tikzcd}$$

- <u>Arnold-Liouville</u>

Any smooth Lag. fib. $\underline{\pi_0 : X_0 \to B_0}$:

a globalization of Log.

$$\begin{tikzcd} (\pi_0)^{-1}(U_i) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Log}^{-1}(V_i) \arrow[d] \\ B_0 \supseteq U_i \arrow[r, "\simeq"] & V_i \subseteq \mathbb{R}^n \end{tikzcd}$$

→ integral affine coordinates

- <u>Kontsevich-Soibelman</u>

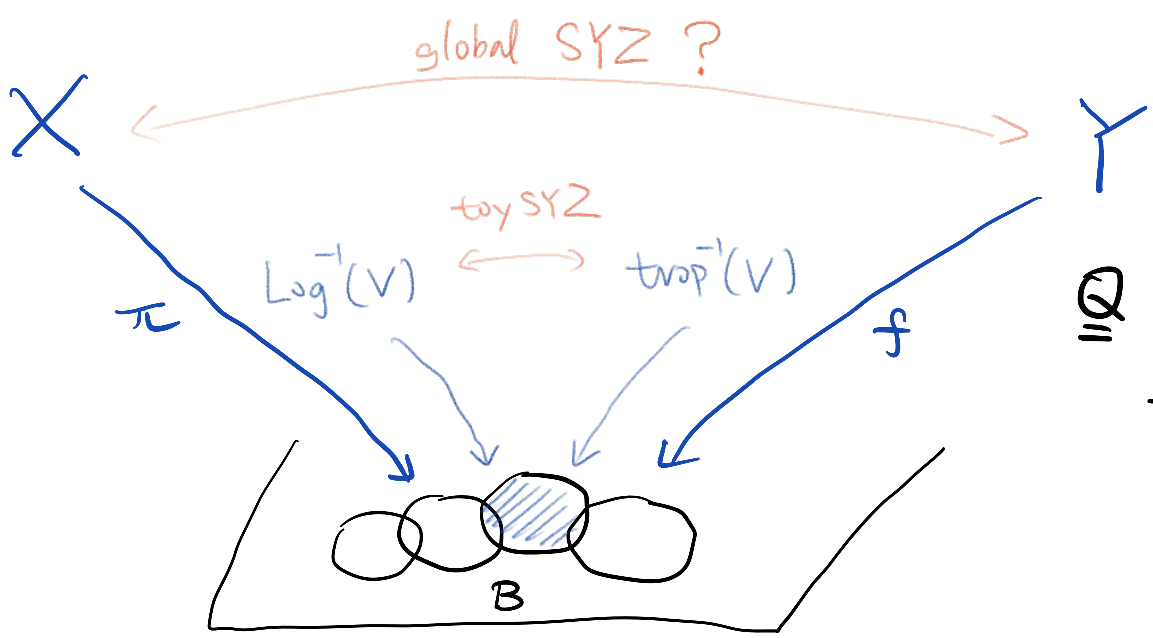
In NA world, <u>a globalization of trop</u> is the notion of affinoid torus fibration.

$f_0 : Y_0 \to B_0$:

Berkovich space

$$\begin{tikzcd} f_0^{-1}(U_i) \arrow[r, "\simeq"] \arrow[d] & \mathrm{trop}^{-1}(V_i) \arrow[d] \\ B_0 \supseteq U_i \arrow[r, "\simeq"] & V_i \subseteq \mathbb{R}^n \end{tikzcd}$$

→ integral affine coordinates!



$\underline{Q}$ given π, find f ?

§ An attempt to precisely formulate ② for SYZ

But we emphasize the starting point is ①

Conj

$$\begin{tikzcd} \underset{\text{Kähler / symplectic}}{X} \arrow[dr, "\pi"'] & & \underset{\text{Berkovich}}{Y} \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

"stable" or special graded Log fib π

f Berkovich-continuous $\cup$ or tropically continuous $\cup$ or ...

s.t.

Today

(i) π and f have the same singular locus Δ

where we cannot make it Log or trop

(ii) $\pi_0 = \pi|_{B_0}$ - - - - - - - -

$f_0 = f|_{B_0}$ is an affinoid torus fibration

~~They induce the same integral affine structure.~~

i.e. same singular integral aff str on B.

(iii) $f_0 \cong \pi_0^\vee \longrightarrow$ the canonical dual affinoid torus fibration

$$\pi_0^\vee : \bigcup_{q \in B_0} H^1(L_q ; U_\Lambda) \longrightarrow B_0$$

with more str

$\boxed{\text{Floer-theoretic Black Box}}$

- We can put a Berkovich space structure sheaf by using the geometry of (X, π_0) My thesis

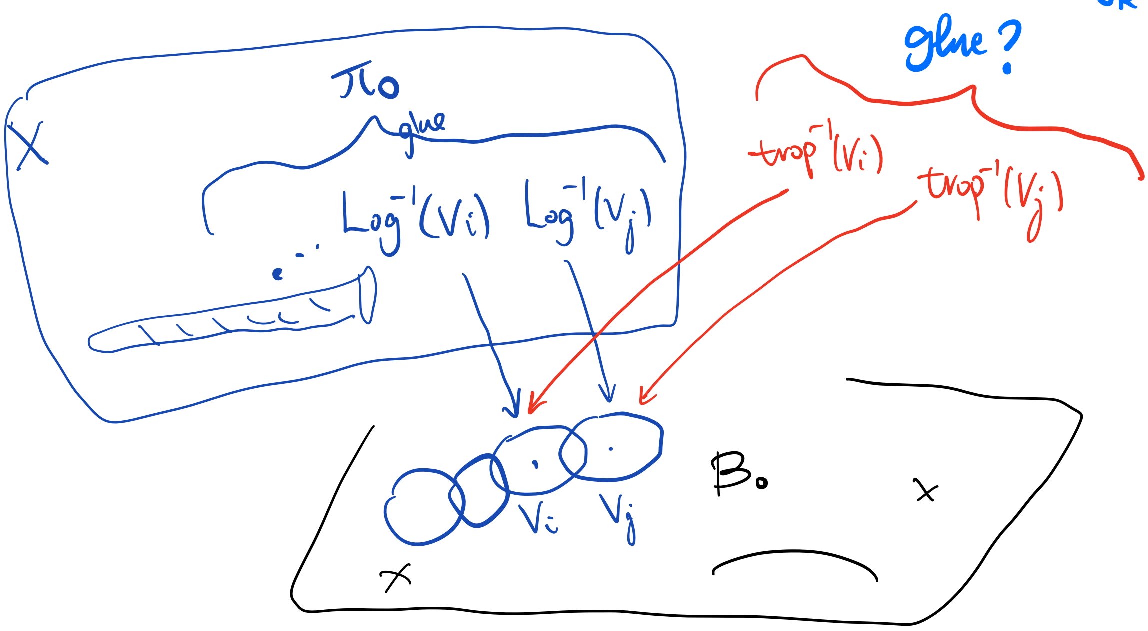
mirror space $\quad X_0^\vee = \bigcup_i \mathrm{trop}^{-1}(V_i)/\sim$

mirror fib $\quad \pi_0^\vee \rightsquigarrow \mathrm{trop}|_{V_i} : \mathrm{trop}^{-1}(V_i) \longrightarrow V_i$

A brief story about (iii)

$y_k \longmapsto T^{c_k} y_k \exp\left(F_k(y_1, \cdots, y_n)\right)$

" A_∞ structure "



- A_∞ algebras from quantum correction holomorphic disks
- rely on FOOO & Kuranishi theory
- However, $\exists$ explicit examples.

§ Examples : outline

X A-side symplectic / Kähler	Y B-side algebraic / Berkovich
positive vertex variety (& analogs)	negative vertex variety. (& analogs)
conifold-smoothing	conifold resolution
A_n - smoothing	A_n - resolution

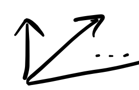
Today

§ Example : A_n sing

$$\boxed{uv = z^{n+1}}$$

smoothing ↙ ↘ resolution

$X = \{uv = (z-a_0)\cdots(z-a_n)\}$ $D = (z=0)$ $\bar{X}$

 $\xrightarrow{\text{fan}} (\Sigma)$

$Y = \mathrm{Toric}(\Sigma) \setminus \tilde{D}$

- $\vec{a} = \{a_0, \cdots, a_n\} \in \mathrm{Conf}_{n+1}$ i.e. $a_i \neq a_j$

- Lagrangian fibration for ω_{std}

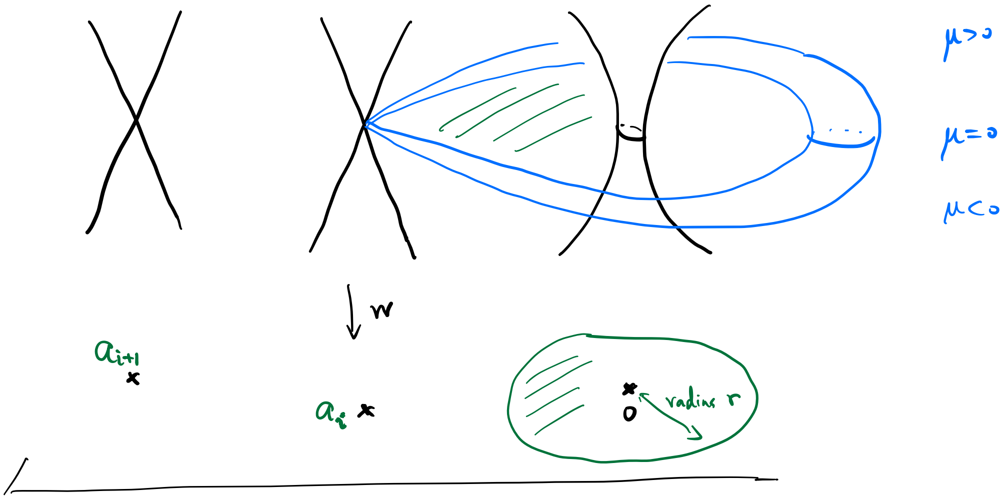
μ: moment map for (ue^{it}, ve^{-it})

$$\pi = \left(\frac{|u|^2 - |v|^2}{2}, |z| \right) \longrightarrow \mathbb{R}^2_{s,r}$$

- Lefschetz fibration $w = z : \bar{X} \longrightarrow \mathbb{C}$

$\forall s \in \mathbb{R}$

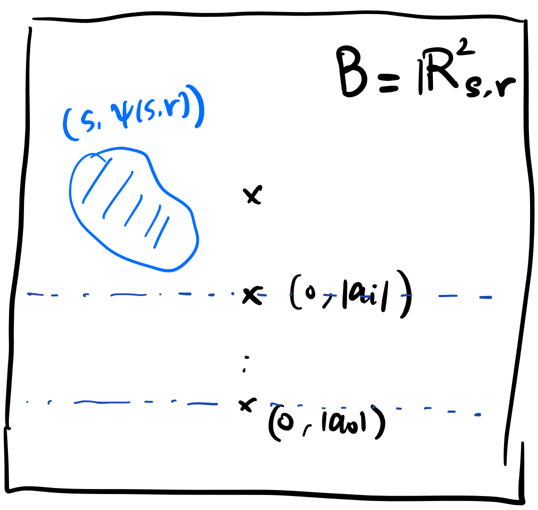
$$\mu^{-1}(s)/S^1 \xrightarrow{\cong} (\mathbb{C}, \omega_{red,s}).$$



$$\boxed{\psi_{\vec{a}}(s,r) \triangleq \int_{D_r} \omega_{red,s}}$$

$\begin{cases} \text{smooth almost everywhere} \\ \text{but non-smooth at } (0, |a_i|) \end{cases}$ $\rightarrow$ mirror NA singular fibers

SYZ base



- $(s, \psi(s,r))$ locally gives integral affine coordinates
- singular points may collide if $|a_i| = |a_j|$

$$X_0^\vee = \bigcup_{i=0}^{n+1} \operatorname{trop}^{-1}(V_i)/\sim$$

Q Want to find a Berkovich-continuous map $f: Y \to B$ with (i), (ii), (iii)

[A meaningful question without Floer theory
But, the solution is from Floer theory

Kontsevich–Soibelman's strategy

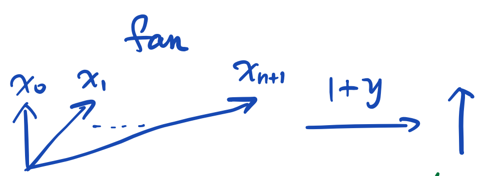
Instead of finding $f: Y \to B$, we aim to find

$F: Y \longrightarrow \mathbb{R}^N$ for $N >> 1$ such that Image $(F) \cong B$.

(ad hoc)

[Any manifold can be embedded in some $\mathbb{R}^N$

- Eventually, find the solution:



Cox's homogeneous coordinates

non-archimedean valuation

$$F([x_0 : x_1 : \cdots : x_{n+1}]) = (F_0, F_1, \cdots, F_{n+1}, v(y)) \text{ in } \mathbb{R}^{n+3}$$

$y = x_0 x_1 \cdots x_{n+1} - 1$

where

disk_k

moment map on A-side.

$$F_k = \left\{ \sum_{j=0}^{n+1} (j-k)\, v(x_j) + k \min\{0, v(y)\},\ \psi_{\vec{a}}(v(y), |a_0|), \right.$$

$$\left. \cdots, \psi_{\vec{a}}(v(y), |a_n|) \right\}_{[k]}$$

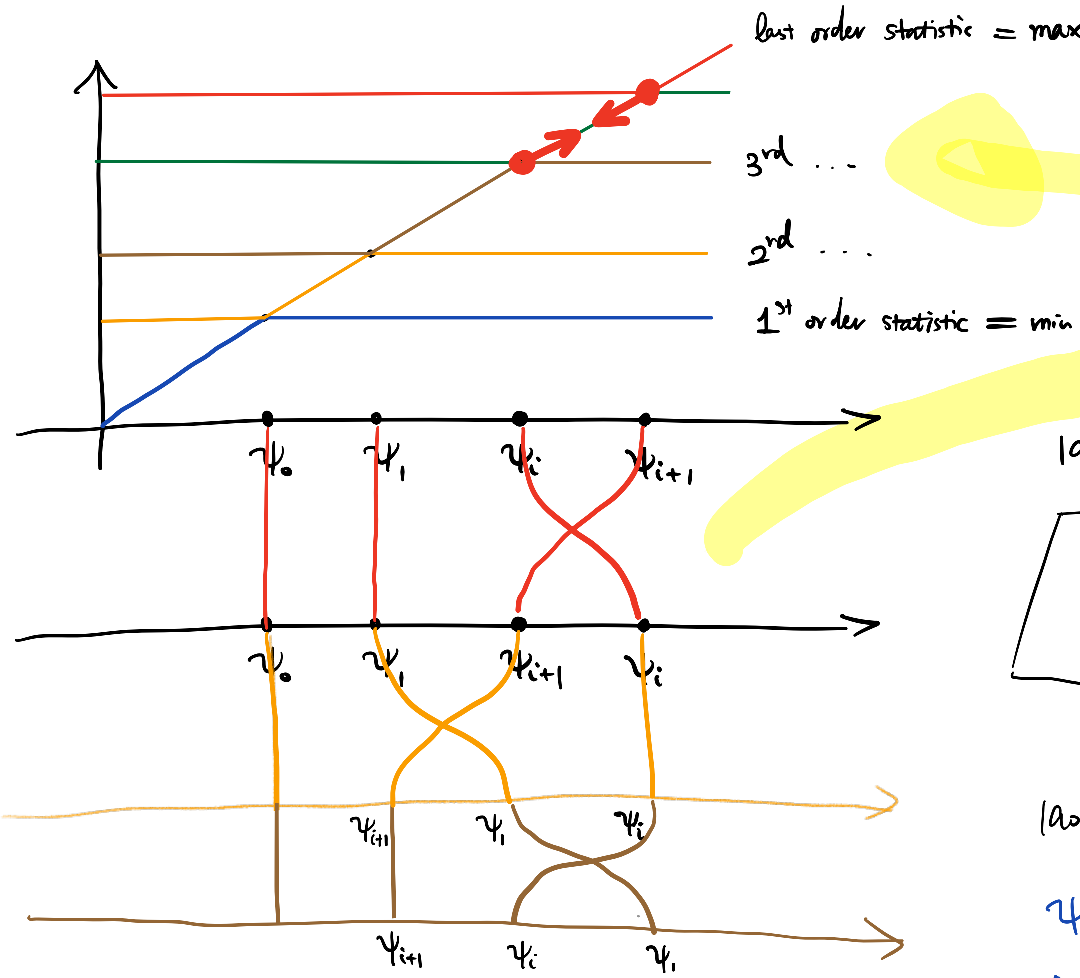
* $\{\cdots\}_{[k]}$ denotes the $(k+1)$-th smallest number among $n+2$ real numbers.

* explicit!

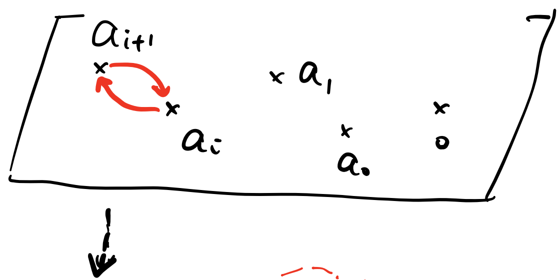
* Use Floer theory to discover it.

Why order statistic?

- locally look like min/max (Kontsevich–Soibelman)
- globally want to detect movemoment of the smoothing parameter $\vec{a} = \{a_0, \dots, a_n\}$.
  - { singular collide / braid group action.



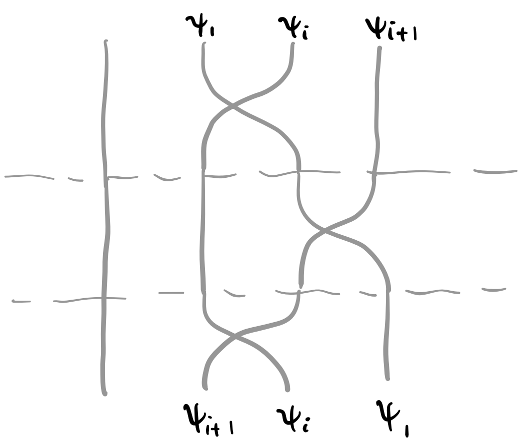
$|a_0| < |a_1| < \cdots < |a_n|$



$|a_0| < \cdots < |a_{i+1}| < |a_i| < \cdots < |a_n|$

$\psi_k = \psi(\bullet, |a_k|)$.

Note $\psi(s, r)$ is increasing in r.



$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$

From order statistic functions,

the relative size among ψ_k's when moving $\vec{a}$

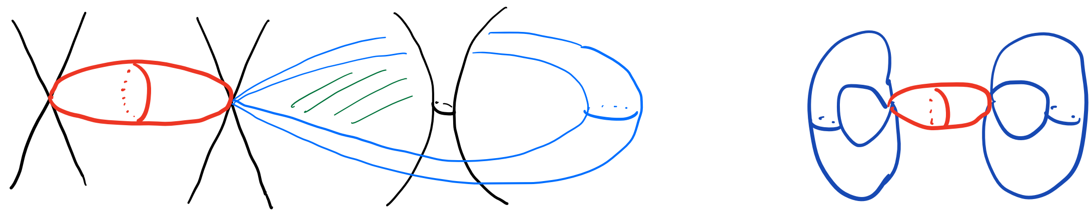
can be detected

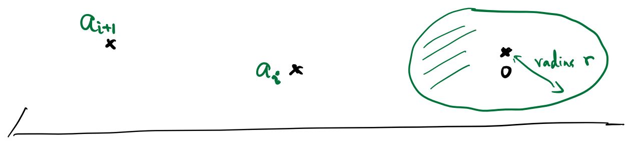
Intuition Size of symplectic areas

$\rightsquigarrow$ Size of non-archimedean valuations

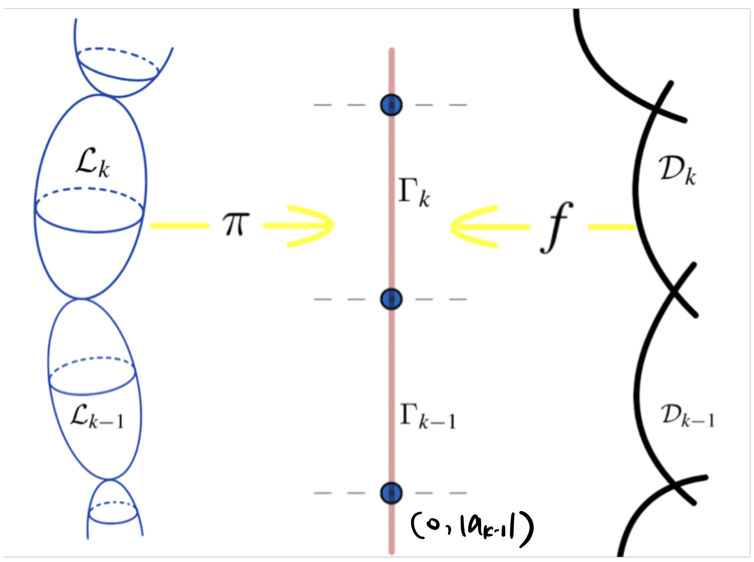
§ A geometric phenomenon

We have Lagrangian spheres on A_n-smoothing





$\mathbb{R}^2_{r,s}$



$$\pi(\mathcal{L}_k) = \Gamma_k = f(\mathcal{D}_k) \quad \cdots\cdots (*)$$

$$\left[\begin{array}{l}\text{Lagrangian vanishing cycles} \\ \text{in } A_n\text{-smoothing}\end{array}\right] \xrightarrow{\pi} \left[\begin{array}{l}\text{tropical lines} \\ \text{connecting focus-focus}\end{array}\right] \xleftarrow{f} \left[\begin{array}{l}\text{exceptional rational} \\ \text{curves}\end{array}\right]$$

- Here we assume $|a_0| < |a_1| < \cdots < |a_n|$ for the above picture.
- But, similar results hold for any $\vec{a} = \{a_0, \cdots, a_n\}$

Future: geometric phenomenon $\overset{?}{\rightsquigarrow}$ categorical result

- Both π and f are explicit!
- Although (π, f) have matchings of integral affine str & singular loci the verification of $(*)$ is merely set-theoretic!
- The matching of integral affine str & singular loci as well as the observation $(*)$ are always true even if we move $\vec{a} = \{a_0, \cdots, a_n\}$.
- $\rightsquigarrow$ strong evidence for SYZ mirror symmetry.

$$\begin{tikzcd} X \arrow[dr, "\pi_{\vec{a}}"'] & & Y \arrow[dl, "f_{\vec{a}}"] \\ & B = \mathbb{R}^2 & \end{tikzcd}$$

(integral affine str $)_{\vec{a}}$

(singular locus $)_{\vec{a}}$

$\vec{a} = \{a_0, \cdots, a_n\} \in \mathrm{Conf}_{n+1}$

Q What happens if we move $\vec{a}$ in a loop in Conf_{n+1}?

Note $\pi_1(\mathrm{Conf}_{n+1}) = \text{Braid gp}$