Open-closed Homotopy Algebra and Relative Disk Mapping space.

(with Yi Wang)

- N = smooth mfd ; $(\Omega(N), d_N, \wedge)$ is a dga.
- $\mathcal{L}N$ = free loop space $\{\gamma : S^1 \to N\}$
- <u>Theorem</u> (K-T Chen and many others)

  If N is simply connected, then there is a natural quasi-isomorphism

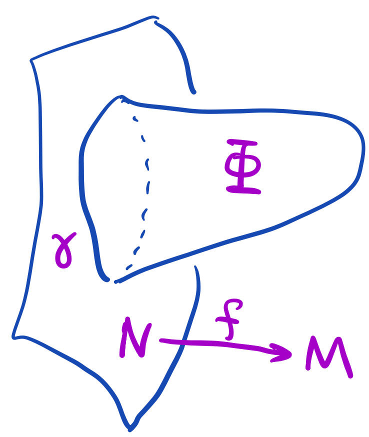
  $$I : C_\bullet(\Omega(N)) \longrightarrow \Omega(\mathcal{L}N)$$

  Hochschild complex of the dga $\Omega(N)$. through the iterated integrals.

§ <u>A relative analogue</u> :

- $f : N \to M$ is a smooth map between smooth manifolds.
- Define

$$\mathcal{X} = \left\{ (\Phi, \gamma) \;\middle|\; \begin{array}{l} \Phi : \mathbb{D} \to M \\ \gamma : S^1 \to N \\ \Phi|_{\partial \mathbb{D}} = f \circ \gamma \end{array} \right\}$$

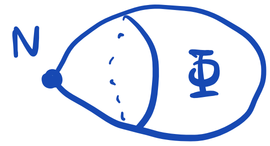


called "relative disk mapping space".

- Just like loop space, $\mathcal{X}$ is also a <u>differentiable space</u>.

- Some special cases

(1) $M = pt$: $f : N \to pt$, $\mathcal{X} = \mathcal{L}N$

(2) $N = pt$: 

$\mathcal{X} \simeq$ based sphere space of M

$\simeq$ double based loop space of M

$\boxed{\text{Goal}}$ Just like how we use the Hochschild homology of the dga $\Omega(N)$ to understand $\Omega(\mathcal{L}N)$ via iterated integrals can we use some "open-closed" version for the pair $(\Omega(M), \Omega(N))$ to understand $\Omega(\mathcal{X})$?

Theorem (Wang - Y)

There is a canonical open-closed iterated integral cochain map

$$J : C_\bullet(\Omega(M) ; \Omega(N)) \longrightarrow \Omega(\mathcal{X})$$

Some open-closed version of Hochschild chain complex

such that

(1) when $M = pt$, the map J recovers $C_\bullet(\Omega(N)) \longrightarrow \Omega(\mathcal{L}N)$

(2) when N is simply connected and when M is contractible or is 2-connected with rational homotopy type of S^{2n-1}, the map J is a quasi-isomorphism.

§ Some homological algebra :

- dga is a special sort of A_∞ algebra ;

  we can actually define Hochschild homology for an A_∞ algebra

  Let's review it

- Consider

$$C^\bullet(A, A) = \prod_k \mathrm{Hom}(A^{\otimes k}, A)$$

on which we have the brace operations :

$$D\{E_1, \cdots, E_n\}(a_1 \cdots a_k) = \sum \pm D(\cdots E_1(\cdots), \cdots, E_n(\cdots) \cdots)$$

<u>Defn</u> An <u>A_∞ algebra</u> on A is $m \in C^\bullet(A, A)$ with $m\{m\} = 0$

$\overset{\shortparallel}{(m_k)_{k \geq 1}}$

- Consider

$$\tilde{C}^\bullet(Z, Z) = \prod_\ell \mathrm{Hom}(Z^{\wedge \ell}, Z)$$

graded symmetric multilinear

$z_1 \wedge z_2 = (-1)^{|z_1||z_2|} z_2 \wedge z_1$

We can similarly define

$$D\langle E \rangle (z_1, \cdots, z_\ell) = \sum_{\substack{\ell_1 + \ell_2 = \ell \\ i_1 < \cdots < i_{\ell_1} \\ j_1 < \cdots < j_{\ell_2}}} \pm D(E(z_{i_1} \wedge \cdots \wedge z_{i_{\ell_1}}) \wedge z_{j_1} \wedge \cdots \wedge z_{j_{\ell_2}})$$

<u>Defn</u> An <u>L_∞ algebra</u> on Z is $L = (L_\ell)_{\ell \geq 1} \in \tilde{C}^\bullet(Z, Z)$ with $L\langle L \rangle = 0$

§ Open-closed homotopy algebra (OCHA)

is a notion introduced by Kajiura-Stasheff 2004

* It is inspired by Zwiebach's open-closed string theory

- Consider

$$C^{\bullet,\bullet}(Z; A, A) = \prod_{l,k} \mathrm{Hom}\left(Z^{\wedge l} \otimes A^{\otimes k}, A\right)$$

Defn An OCHA is the data (Z, L, A, q)

where (Z, L) is an L_∞ algebra

and $q = (q_{l,k}) \in C^{\bullet,\bullet}(Z; A, A)$ such that

$$\sum \pm q\left(z_{i_1} \wedge \cdots \wedge z_{i_{l_1}} ; a_1 \cdots q(z_{j_1} \wedge \cdots z_{j_{l_2}} ; a_\cdot \cdots) \cdots a_k\right)$$

$$= \sum \pm q\left(L(z_{i_1} \wedge \cdots \wedge z_{i_{l_1}}) \wedge z_{j_1} \wedge \cdots \wedge z_{j_{l_2}} ; a_1 \cdots a_k\right)$$

Remark The sub-collection $\{q_{0,k}\}_k$ is precisely an A_∞ algebra.

It's somehow a joint generalization of both A_∞ algebra and L_∞ algebra.

§ Open-closed Hochschild Cohomology

Given an OCHA (Z, A, l, q), we can define a differential δ on $C^{\bullet,\bullet}(Z; A, A)$ by

$$\delta(D)(z_1 \cdots z_l ; a_1 \cdots a_k)$$

$$= \sum \pm q(z_{i_1} \cdots z_{i_{l_1}} ; a_1 \cdots D(z_{j_1} \cdots z_{j_{l_2}} ; a_\lambda \cdots a_\mu) \cdots a_k)$$

$$+ \sum \pm D(z_{i_1} \cdots z_{i_{l_1}} ; a_1 \cdots q(z_{j_1} \cdots z_{j_{l_2}} ; a_\lambda \cdots a_\mu) \cdots a_k)$$

$$+ \sum \pm D(l(z_{i_1} \cdots z_{i_{l_1}}) \wedge z_{j_1} \cdots z_{j_{l_2}} ; a_1 \cdots a_k).$$

$\delta^2 = 0 \implies$ We can define Hochschild cohomology

$$HH(Z; A, A) = H(C^{\bullet,\bullet}(Z; A, A), \delta)$$

Theorem $HH(Z; A, A)$ has a canonical Gerstenhaber algebra structure. Moreover, if $(Z, A; l, q)$ is cyclic, then $HH(Z; A, A)$ is a BV algebra.

Similar to cyclic A_∞ algebra

§ Open-Closed Hochschild Homology

- Fix a general OCHA $(Z, A, \mathfrak{l}, \mathfrak{q})$. (unital OCHA)
- Define the open-closed Hochschild chain complex as

$$C_{\bullet,\bullet}(Z;A) = \bigoplus C_{\ell,k}(Z;A)$$
$$= \bigoplus \bar{Z}^{\wedge \ell} \otimes A \otimes \bar{A}^{\otimes k}$$

equipped with the differential map

$$b = b_0 + b_1 + b_2 : C_{\bullet,\bullet}(Z;A) \circlearrowleft$$

where

$$b_0(z_1 \wedge \cdots \wedge z_\ell \otimes a_0 \otimes a_1 \cdots a_k)$$
$$= \sum \pm \mathfrak{l}(z_{i_1} \wedge \cdots \wedge z_{i_{\ell_1}}) \wedge z_{j_1} \wedge \cdots \wedge z_{j_{\ell_2}} \otimes a_0 \cdots a_k$$

$$b_1 = \sum \pm z_{i_1} \wedge \cdots \wedge z_{i_{\ell_1}} \otimes \underline{a_0} \cdots \mathfrak{q}(z_{j_1} \wedge \cdots \wedge z_{j_{\ell_2}}; a_\lambda \cdots a_\mu) \cdots a_k$$

$$b_2 = \sum \pm z_{i_1} \wedge \cdots \wedge z_{i_{\ell_1}} \otimes \mathfrak{q}(z_{j_1} \wedge \cdots \wedge z_{j_{\ell_2}}; a_{\mu+1} \cdots a_k \underline{a_0} a_1 \cdots a_\lambda) a_{\lambda+1} \cdots a_\mu$$

- We can also define an open-closed variant of Connes B-operator

$$B(z_1 \wedge \cdots \wedge z_\ell \otimes a_0 \cdots a_k) = \sum \pm z_1 \wedge \cdots \wedge z_\ell \otimes 1_A \otimes a_{i+1} \cdots a_k a_0 \cdots a_i$$

<u>Prop</u> $b^2 = B^2 = bB + Bb = 0$.

<u>Defn</u> Open-closed Hochschild homology is defined as

$$HH(Z; A) = H(C_{\cdot,\cdot}(Z; A), b)$$

(We don't discuss B today)

- Now, go back to geometry:

OCHA

§ Example from the smooth map $f: N \longrightarrow M$

$$\left\{ \begin{array}{l} Z = \Omega(M)[2] \quad ; \quad A = \Omega(N)[1] \\ \quad \omega \in \Omega(M) \quad \text{shift degree} \\ \quad |\omega| = \deg_{dR} \omega - 2 \\ L_1 = d_M . \quad L_\ell = 0 \text{ for } \ell \geq 2 \\ q_{1,0} = f^* . \quad \Omega(M) \longrightarrow \Omega(N) \\ q_{0,1} = d_N \quad ; \quad q_{0,2} = \pm \wedge_N \end{array} \right.$$

- One can check these give an OCHA
- Somehow it is like a relative version of dga
- Our main theorem about $J : C(\Omega(M); \Omega(N)) \longrightarrow \Omega(\mathcal{X})$

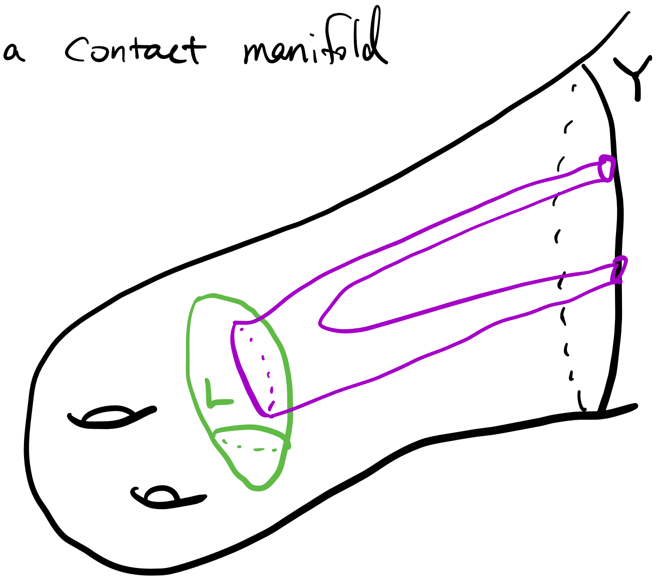
(open-closed) Hochschild chain cx of this OCHA

§ Potential OCHA Example in Symplectic Field Theory.

- X is an open symplectic manifold with

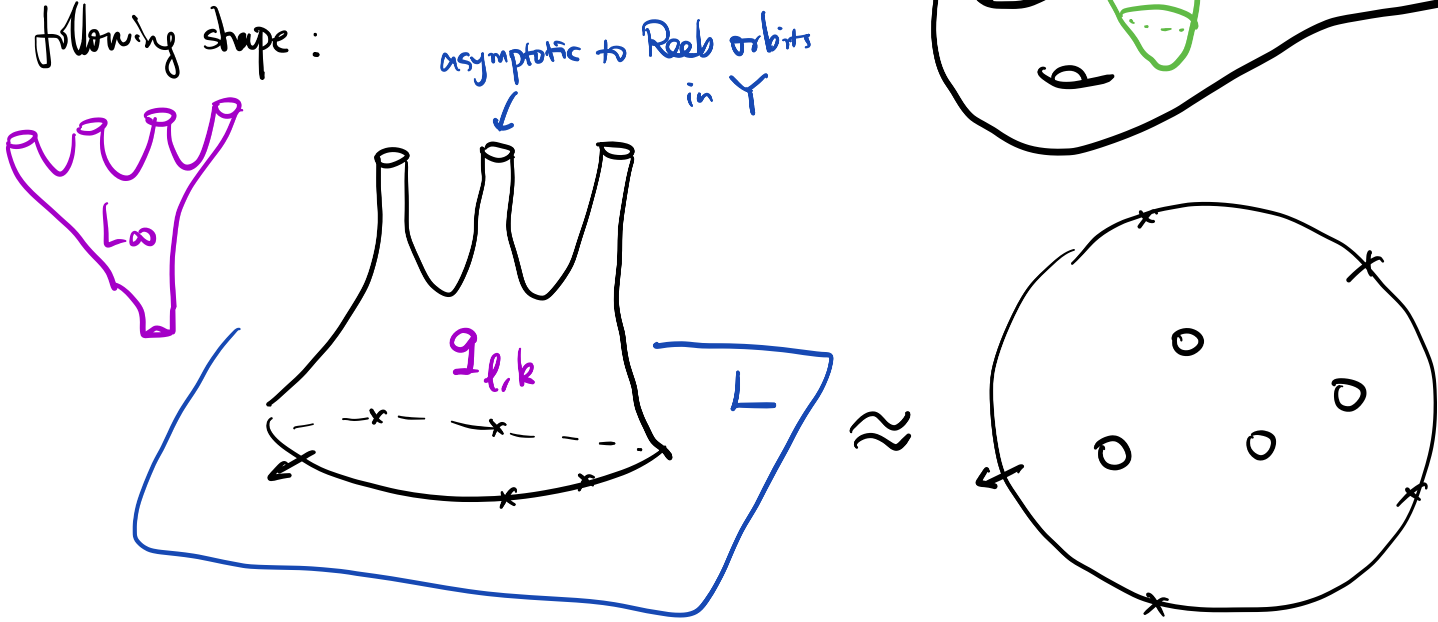
$$X = K_X \cup [0,\infty) \times Y$$

where K_X is a compact domain and Y is a contact manifold



- $L \subseteq K_X$ is a Lagrangian submfd

- moduli of pseudo-holo curves of the following shape :



- This is an analogue of Fukaya-Oh-Ohta-Ono's operator $\mathfrak{q}$ introduced in their Lagrangian Floer theory.

- The transversality for this moduli is in progress (FOOO's Kuranishi structure)

§ Open-closed iterated integral map

→ use the previous example of OCHA

$$\mathcal{J} : C_{\bullet,\bullet}\big(\Omega(M)[2],\ \Omega(N)[1]\big) \longrightarrow \Omega(\mathcal{X})$$

$\boxed{\text{Claim}}$ The map $\mathcal{J}$ can be naturally defined for any arbitrary smooth map $f : N \longrightarrow M$.

However, to guarantee $\mathcal{J}$ is a quasi-isomorphism, we must make some extra assumptions on $f : N \longrightarrow M$.

Recall
$$\mathcal{X} = \left\{ (\Phi, \gamma) \;\middle|\; \begin{array}{l} \Phi : \mathbb{D} \longrightarrow M \\ \gamma : S^1 \longrightarrow N \\ \Phi(e^{2\pi i t}) = f(\gamma(t)) \end{array} \right\}$$

$$\Delta^k = \{ 0 \leq t_1 \leq \cdots \leq t_k \leq 1 \} \qquad k\text{-simplex}$$

Consider the evaluation map

$$Ev_{\ell,k} : \mathbb{D}^\ell \times \Delta^k \times \mathcal{X} \longrightarrow M^\ell \times N^{k+1}$$

$(z_1, \cdots, z_\ell, t_1, \cdots, t_k, (\Phi, \gamma))$

$$\longmapsto \left(\Phi(z_1), \cdots \Phi(z_\ell), \gamma(0), \gamma(t_1), \cdots, \gamma(t_k) \right)$$

Define $J = (J_{\ell,k})$ with

$$J_{\ell,k} : \Omega(M)^{\otimes \ell} \otimes \Omega(N)^{\otimes k+1} \longrightarrow \Omega(M^\ell \times N^{k+1})$$

$$\xrightarrow{Ev_{\ell,k}^*} \Omega(\mathbb{D}^\ell \times \Delta^k \times \mathcal{X})$$

$$\xrightarrow[\text{fibers}]{\text{"integration along"}} \Omega^{\bullet - 2\ell - k}(\mathcal{X})$$

Theorem $d_{\mathcal{X}} \circ J = J \circ b$

(open-closed) Hochschild chain complex.

<u>Sketch of idea</u>:

The boundary components of $Q_{\ell,k} := \mathbb{D}^\ell \times \Delta^k$

is as follows

$$\partial Q_{\ell,k} \overset{\text{"up to sign"}}{\cong} \partial \mathbb{D}^{\ell} \times \Delta^{k} \ \sqcup \ \mathbb{D}^{\ell} \times \partial \Delta^{k}$$

$$\cong \bigcup_{r} \mathbb{D}^{r-1} \times S^{1} \times \mathbb{D}^{\ell-r} \cup \bigcup_{j} \mathbb{D}^{\ell} \times \underline{j\text{th facet of } \Delta^{k}}$$

We also develop "integration along fiber" for differentiable space of the form $Q \times \mathcal{X}$

where Q is a finite-dimensional smooth manifold and $\mathcal{X}$ is a general differentiable space (loop space, disk space)

- $\int_{Q} : \Omega(Q \times \mathcal{X}) \longrightarrow \Omega(\mathcal{X})$

- $i_{Q}^{*} : \Omega(Q \times \mathcal{X}) \longrightarrow \Omega(\partial Q \times \mathcal{X})$

$$\Rightarrow (-1)^{\bullet} d_{\mathcal{X}} \circ \int_{Q} = \int_{Q} \circ d_{Q \times \mathcal{X}} - \int_{\partial Q} \circ i_{Q}^{*}$$

§ Idea for Quasi-isomorphism results for J

Theorem (Wang–Y)

There is a canonical open-closed iterated integral cochain map

$$J : C_\bullet(\Omega(M) ; \Omega(N)) \longrightarrow \Omega(\mathcal{X})$$

Some open-closed version of Hochschild chain complex

such that

(1) when $M = pt$, the map J recovers $C_\bullet(\Omega(N)) \longrightarrow \Omega(LN)$

(2) when N is simply connected and when M is contractible or is 2-connected with rational homotopy type of S^{2n-1}, the map J is a quasi-isomorphism.

- Consider the following fibration

$$\mathcal{X} = \mathrm{Map}((D, S^1), (M, N))$$

$$\mathcal{X}_0 = \mathrm{Map}((D, S^1), (M, pt))$$

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r] & \mathcal{X} \arrow[d] & (\Phi, \gamma) \arrow[d, mapsto] \\ & LN & \gamma \end{tikzcd}$$

- Actually, we have the following pullback diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathrm{Map}(\mathbb{D}, M) \arrow[d] & \Phi \arrow[d, mapsto] \\
\mathcal{L}N \arrow[r, "\mathcal{L}f"] & \mathcal{L}M & \Phi|_{\partial \mathbb{D}} \\
\gamma \arrow[r, mapsto] & f \circ \gamma &
\end{tikzcd}$$

- Roughly speaking, our idea is to study Serre-type spectral sequence for the fibration $\mathcal{X}_0 \hookrightarrow \mathcal{X} \longrightarrow \mathcal{L}N$, $(\Phi, \gamma) \mapsto \gamma$

- The models for $\mathcal{X}_0$ and $\mathcal{L}N$ are largely known

- We also need a filtration on the open-closed Hochschild chain complex $\left(C_\cdot(\Omega(M); \Omega(N)),\ b \right)$

which is similar to the filtration on the usual Hochschild chain $(C_\cdot(\Omega(N)), b)$

§ Further Directions

- Open-closed Cyclic homology :

* We also have Connes operator B corresponding to the S^1-action inherited from the boundary loop

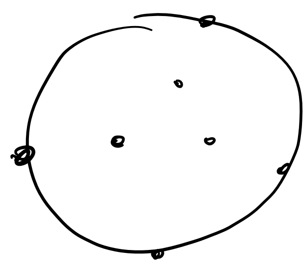
* It should be dual to the BV operator Δ for the BV algebra str on open-closed Hochschild Cohomology

* $H_*(\mathcal{H})$ similar to $H_*(\mathcal{L}N)$

$\Updownarrow$

open-closed Hochschild cochain complex (following K. Irie and Yi Wang's work)

- Weaker Assumption on M



$\rightsquigarrow$ relative configuration space

v.s. $\mathrm{Conf}_\ell(\mathbb{C})$

$$= \{ (z_1, \dots, z_\ell) \in \mathbb{C}^\ell \mid z_i \neq z_j \}$$

? $\mathrm{Conf}_{\ell,k}(\mathbb{D}, \partial\mathbb{D}) \overset{?}{=} \{ (z_1, \dots, z_\ell, w_1 \dots w_k) \in \mathbb{C}^\ell \times \mathbb{D}^k$

$\mid$ pairwise distinct $\}$