Existence of pseudo-holomorphic disks via non-archimedean disk potentials

?

§ Review: disk potential for monotone Lagrangian $L \subseteq X$

- A well-known enumerative invariant for a monotone Lag $L \subseteq X$ is the algebraic count of Maslov-2 pseudo-holomorphic disks bounded by L (Eliashberg-Polterovich 93, 97, Auroux, Vianna, ...)

- $\beta \in H_2(X, L)$ with Maslov index $\mu(\beta) = 2$; J ω-tame almost cx str.

- Consider the moduli space of J-holomorphic disks with one boundary marked pt

- $$\mathcal{M}_1(L, \beta, J) = \left\{ u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L) \,\middle|\, \bar{\partial}_J u = 0 , \ [u] = \beta \right\} \Big/ \mathrm{Aut}(\mathbb{D}, 1)$$

  dimension $= n + \mu(\beta) - 2 = n = \dim(L)$.

- The degree of the evaluation map (J generic)

$$ev : \mathcal{M}_1(L, \beta, J) \longrightarrow L \ , \ [u] \mapsto u(1)$$

gives a number $\boxed{n_\beta}$: "signed count of disks in the class β"

- Assemble them into the disk potential of a monotone Lagrangian L :

$\to \int_\beta \omega$ symplectic energy

①

$$W_L = \sum_{\substack{\beta \in H_2(X,L) \\ \mu(\beta)=2}} n_\beta \, e^{-E(\beta)} \underbrace{hol_\nabla(\partial\beta)}_{\text{holonomy}}$$

∇ unitary flat connection

Remark. • In SYZ mirror symmetry, this W_L corresponds to the Landau Ginzburg mirror superpotential on a mirror space. cf. (Auroux) T-duality paper

Question: generalization of W_L to non-monotone Lagrangian L?

Fukaya-Oh-Ohta-Ono :

$$② \qquad W_L : \underbrace{MC(L)}_{\text{Maurer-Cartan } \underline{\underline{\text{set}}}\text{, i.e. the set of weak bounding cochains}} \longrightarrow \overset{\text{Novikov field/ring}}{\Lambda}$$

- This setting works for very general Lagrangian.

- However, the setting of ① also has its own advantages.

a Laurent polynomial

- Because the W_L in ① can be viewed as $\boxed{\text{a function}}$ on a spac

, while $MC(L)$ in ② is often just a set.

Ex For instance, Vianna uses the Newton polytope of W_L in ① to produce some applications.

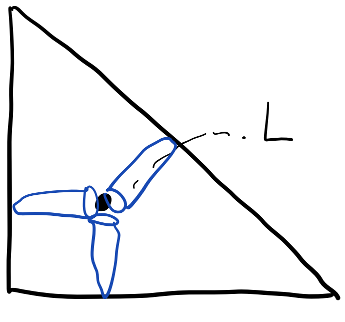
not make sense for ②

- Concretely, if we choose a basis $\gamma_1, \cdots, \gamma_n$ of $H_1(L)$ and write $y_i = \mathrm{hol}_\nabla(\gamma_i) \in \mathbb{C}^*$, $\partial\beta = \partial_1\beta \cdot \gamma_1 + \cdots + \partial_n\beta\, \gamma_n$ then ① becomes a Laurent polynomial:

$$W_L(y_1, \cdots, y_n) = \sum_{\mu(\beta)=2} e^{-E(\beta)} n_\beta\, y_1^{\partial_1\beta} \cdots y_n^{\partial_n\beta} : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

$$= \sum T^{E(\beta)} y^{\partial\beta} n_\beta \qquad \left(T = e^{-1} \right)$$

Ex If $X = \mathbb{CP}^2$ and $L \subseteq X$ central moment map fiber



$$W_L = y_1 + y_2 + \frac{c}{y_1 y_2} \quad \text{for some } c$$

Goal Now, when L is not necessarily monotone, we aim to build an intermediate framework between ① and ② so that W_L can be still viewed as a function on a space

in what category of spaces?

Application:

The well-definedness of such generalized disk potential $\Longrightarrow$

Theorem Let $\{L_s : s \in [0,1]\}$ be a smooth isotopy of graded Lagrangian submanifolds. Assume L_0 is monotone and $\underbrace{W_{L_0} \neq 0}_{\text{classical}}$. Then, for every s, $\underbrace{W_{L_s} \neq 0}_{\text{to be defined}}$ and for every ω-tame J, there exists a Maslov-2 J-holo disk bounded by L_s.

Further Application

Theorem (Rizell-Goodman-Ivrii) If (X,ω) is one of $\mathbb{C}^2$, $\mathbb{CP}^2$, $S^2\times S^2$, any two Lagrangian torus in X are Lagrangian isotopic to each other.

Corollary In this case, for every ω-tame J, every Lagrangian torus $L\subseteq X$ bounds a Maslov-2 J-holomorphic disk.

- In particular, for $X=\mathbb{C}^2$, we retrieve a basic case of Audin's conjecture (Viterbo, Polterovich)

Now, go back to our goal:

§ Generalization of disk potential.

Issue W_L may contain infinitely many terms.

e.g. $\alpha \in H_2(X, L)$ $\quad \mu(\alpha) = 0$ — Maslov-zero disk.

$\beta \in H_2(X, L)$ $\quad \mu(\beta) = 2$ $\quad \Rightarrow \mu(\beta + k\alpha) = 2$ for $\forall k \in \mathbb{Z}$

- First, we should work with the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; \lambda_i \in \mathbb{R}, \quad \lambda_i < \lambda_{i+1} < \cdots \to \infty \right\}$$

- Since $W_L : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ in the monotone case, one may expect that in general, we have: $\quad W_L : (\Lambda^*)^n \longrightarrow \Lambda$

$$\rotatebox{90}{$\subseteq$}$$

- However, we actually only have $\quad W_L : \underbrace{\operatorname{trop}^{-1}(V)}_{\text{analytic open subset (Berkovich geometry)}} \longrightarrow \Lambda$

due to the convergence issue.

- We will define the (generalized) disk superpotential as a functor

$$\begin{Bmatrix} A_\infty \text{ algebras} \\ \text{with certain unobstructed conditions} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{analytic functions on} \\ \text{some domain of } (\Lambda^*)^n \end{Bmatrix}$$

§ Input : Fukaya's A_∞ algebras

$L \subseteq X$ a Lagrangian $\left\{ \begin{array}{l} \text{not necessarily monotone} \\ \text{(relatively) spin, closed, oriented, etc.} \cdots \end{array} \right.$

$\beta \in H_2(X, L)$; J ω-tame almost complex structure

$$\mathcal{M}_{k+1,\beta}(J, L) = \left\{ (\Sigma, u, z_0, z_1, \dots, z_k) \;\middle|\; \begin{array}{l} \Sigma \text{ nodal curve with boundary} \\ z_i \in \partial\Sigma \\ u : (\Sigma, \partial\Sigma) \longrightarrow (X, L) \\ \qquad\qquad J\text{-holomorphic} \end{array} \right\} \Big/ \sim$$

- Roughly speaking, with some extra choice, the moduli space gives an operation

$$\begin{tikzcd} & \mathcal{M}_{k+1,\beta}(J, L) \arrow[dl, "{ev_1, \dots, ev_k}"'] \arrow[dr, "ev_0"] & \\ \underbrace{L \times \cdots \times L}_{k} & & L \end{tikzcd}$$

$$m_{k,\beta} : \Omega^*(L)^{\otimes k} \longrightarrow \Omega^*(L)$$

$$(h_1, \dots, h_k) \longmapsto \text{“}ev_{0!}\text{”}\left(ev_1^* h_1 \wedge \dots \wedge ev_k^* h_k\right)$$

Kuranishi structure (Fukaya-Oh-Ohta-Ono)

- ## Axioms for Fukaya's A_∞ algebras

(a) Fix J. There exists a set $V(J)$ of "virtual fundamental chains"

such that any element Ξ

is associated with an A_∞ algebra $m^{J,\Xi}$

partial topology (Riemannian metrics)

Kuranishi structure (obstruction bundle data)

CF-perturbation data ...

(b) For different choices J, Ξ ; J', Ξ',

there is a pseudo-isotopy between $m^{J,\Xi}$ and $m^{J',\Xi'}$

gives an A_∞ homotopy equivalence

- Using the homological perturbation, we can make $m^{J,\Xi}$ an A_∞ algebra on $H^*(L)$ instead of $\Omega^*(L)$.

- Given $m := m^{J,\Xi}$, we set

formal power series group ring

$$W_m = \sum_{\substack{\beta \in H_2(X,L) \\ \mu(\beta)=2}} T^{E(\beta)} Y^{\partial\beta} m_{0,\beta} \in \Lambda[[\pi_1(L)]]$$

- Here $m_{0,\beta} \in H^{2-\mu(\beta)}(L)$. If $\mu(\beta)=2$, then $m_{0,\beta}$ is a number.
  This precisely mimics the monotone case with $n_\beta \longleftrightarrow m_{0,\beta}$

$$e^{-E(\beta)} \longleftrightarrow T^{E(\beta)}$$

$$\mathrm{hol}_\nabla(\partial\beta) \longleftrightarrow y^{\partial\beta}$$

- We require that the Maslov index $\mu(\beta) \geq 0$ if some $[u] = \beta$.
  So, $m_{0,\beta} \in H^{2-\mu(\beta)}(L)$ $\begin{cases} \text{either } H^0 \\ \text{or } H^2. \end{cases}$

- Unlike the monotone case, this is not invariant and depends on extra auxiliary choices.

- However, with certain unobstructedness condition, we can make W_m invariant up to analytic isomorphism.

we propose :

$$Q_m := \sum_{\mu(\beta)=0} T^{E(\beta)} y^{\partial\beta} \underbrace{m_{0,\beta}}_{\in H^2(L)}$$

Observation Up to a coordinate change, a solution to Q_m is essentially a weak bounding cochain. (Using divisor axiom transfers Q_m to Maurer-Cartan eq.) If it exists, then we say the A_∞ algebra m unobstructed in the literature

Defn We say the A_∞ algebra m "strongly unobstructed" if $Q_m \equiv 0$

Idea

$\exists$? individual solution to the Maurer-Cartan equation $\rightsquigarrow$ View the Maurer-Cartan equations as analytic functions.

Remark This is much stronger but reasonable since monotone Lagrangians are strongly unobstructed. (no Maslov-0 disks)

Claim With "strong" unobstructedness assumption, we have

$$\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, "f"', "A_\infty \text{ homotop equiv.}"] & W_m \arrow[d, "\Phi_f : y_k \mapsto y_k \exp F_k(y_1 \cdots y_n)"] \\
m' \arrow[r, mapsto] & W_{m'}
\end{tikzcd}$$

$\boxed{W_m \text{ is well-defined up to analytic isom.}}$

$\mathcal{G} \,//\, \mathrm{Aut}_{\mathbb{Z}}(\mathcal{G})$

§ Further direction

(1) In the monotone case, Vianna studies the Newton polytope of the disk potential (Laurent polynomial) to distinguish Lagrangian tori.

- In the non-archimedean geometry, Einsiedler-Kapranov-Lind 2006 introduced the extended Newton polytope

$$f = \sum_{\nu\in\mathbb{Z}^n} a_\nu y^\nu \quad , \quad y^\nu = y_1^{\nu_1}\cdots y_n^{\nu_n}.$$

$a_\nu \in \Lambda$ Novikov

Convex hull of $\left\{ (\nu, s) \in \mathbb{Z}^n \times \mathbb{R} \;\middle|\; s \geq \mathrm{val}(a_\nu) \right\}$

Its projection is precisely the Newton polytope

$$\text{Symplectic geometry} \begin{cases} \text{rigidity} \overset{?}{\longleftrightarrow} \text{non-archimedean analytic structure.} \\ \text{flexibility} \end{cases}$$

§ A quick review of Berkovich geometry

- Roughly speaking, we can do analytic geometry over Novikov field (non-archimedean)
- $(k, |\cdot|)$ is a non-archimedean field. $|x+y| \leq \max\{|x|, |y|\}$
- Tate algebra $$k\langle T_1, \dots, T_n\rangle = \left\{ \sum_{\nu \in \mathbb{Z}^n_{\geq 0}} a_\nu T_1^{\nu_1} \cdots T_n^{\nu_n} \;\middle|\; \begin{array}{l} a_\nu \in \Bbbk \\ \lim\limits_{|\nu| \to \infty} |a_\nu| = 0 \end{array} \right\}$$
- affinoid algebra $$A \cong k\langle T_1, \dots, T_n\rangle / I$$
- The Berkovich spectrum

$$\mathcal{M}(A) = \text{the set of multiplicative seminorms on } A$$

extending the non-archimedean norm on k.

Remark $A = k\langle T_1, \dots, T_n\rangle$. Then

$$\mathrm{MaxIdeal}(A) \cong \{ x = (x_1, \dots, x_n) \in k^n \mid |x_i| \leq 1 \}$$

$\rotatebox[origin=c]{-90}{$\subseteq$}$

$\mathcal{M}(A)$ add more "generic points"

Such a point $x = (x_1, \dots, x_n)$ defines a seminorm

$$|\cdot|_x : A \longrightarrow \mathbb{R}_{\geq 0}, \quad f = \sum a_\nu T^\nu \longmapsto |f(x)|$$

- <u>Defn</u> A (Berkovich) analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that $\forall\, x \in X$ admits $\begin{cases} \text{an open neighborhood } U \\ \text{an affinoid algebra } A_U \end{cases}$

with

$$(U, \mathcal{O}_X|_U) \cong \mathcal{M}(A_U)$$

- In brief, the basic building blocks are <u>affinoid algebras</u> and their morphisms.

- Analytification of algebraic variety $X \rightsquigarrow X^{an}$.

- $\Lambda^* = \Lambda \setminus \{0\}$. $(\Lambda^*)^n = \operatorname{Spec} \Lambda[y_1^{\pm}, \cdots, y_n^{\pm}] \rightsquigarrow (\Lambda^*)^{n, an}$

- Explicitly, $(\Lambda^*)^{n,an}$ = the set of multiplicative seminorms on $\Lambda[y_1^{\pm} \cdots y_n^{\pm}]$ and every point $\underbrace{x = (x_1, \cdots, x_n) \in (\Lambda^*)^n}_{\text{called classical points}}$ defines such a seminorm.

(Type I points)

- The tropicalization map

$$\mathrm{trop} : (\Lambda^*)^{n,an} \longrightarrow \mathbb{R}^n$$

$$(x_1, \cdots, x_n) \longmapsto (-\log|x_1|, \dots, -\log|x_n|)$$

$$\text{i.e.} \quad (x \leftrightarrow |\cdot|_x) \longmapsto (-\log|y_1|_x, \cdots, -\log|y_n|_x)$$

It is a continuous, surjective, and proper map with respect to the Euclidean topology and the analytic topology on $(\Lambda^*)^{n,an}$.

- Let $Z = \mathrm{trop}^{-1}(0)$ and let

$$\mathcal{G} = \varinjlim_{V \ni 0} \mathcal{O}(\mathrm{trop}^{-1}(V))$$

be the ring of germs of analytic functions on Z.

Then the previous $W_m \in \mathcal{G}$